Online Assessments and Interactive Classroom Sessions:

A Potent Prescription for Ailing Success Rates in Social Science Calculus

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Mot-clés: Enseignement collégial, Mathématique, Apprentissage, Persévérance, Connaissance de la propre capacité

Au cours des dix dernières années, l'inscription et la réussite des étudiants au cours de calcul différentiel du programme de sciences humaines ont grandement diminué. La présente étude à pour but de déterminer la réversibilité de cette tendance, principalement avec l'ajout d'une session interactive et de devoirs en ligne. WeBWorK est un atout à la pédagogie et un système de devoirs en ligne qui sera utilisé en classe. Dans le cadre de ce projet, trois stratégies pédagogiques ont été développées et évaluées: (C1) - lectures traditionnelles jumelées à des devoirs soumis aux professeurs; (C2) - lectures traditionnelles jumelées à des devoirs en ligne WeBWorK et enfin (C3) - lectures traditionnelles jumelées à des sessions de classe interactives conçues pour épauler le professeur et les étudiants travaillant sur leurs devoirs en ligne à partir de WeBWorK. Dans cette étude quasi-expérimentale, le rendement scolaire des étudiants, leur persévérance aux cours de mathématiques, leur connaissance de leurs propres capacités et leur motivation sont utilisés comme critères d'évaluation.

Le rendement scolaire des étudiants avait été évalué au moyen des notes finales et de leur niveau de connaissance de la matière au cours de calcul différentiel. La connaissance de la matière avait été mesurée par le biais d'un codage des travaux des étudiants au moyen de codeurs indépendants. Nous avons également évalué les connaissances en algèbre et en fonctions des étudiants, acquises auparavant, déterminant ainsi si elles influencent ou non leur apprentissage du calcul différentiel. Quatre questions ont été soulevées dans le cadre de cette recherche :

1. Concernant les étudiants mal préparés : Leur mauvaise préparation a-t-elle un impact significatif sur leur apprentissage du cours de calcul différentiel?

   Pour répondre à cette question, nous avons développé et validé, de façon intrinsèque et extrinsèque, deux mesures de connaissances en algèbre et en fonctions. Ces mesures pourraient bien devenir des outils importants pour aider les professeurs à évaluer le niveau de connaissances de leurs étudiants. Par conséquent, elles peuvent aussi contribuer à munir

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les professeurs de cette information, leur permettant ainsi d'aider les étudiants ayant des lacunes à développer des stratégies de calcul. À l'aide de ces mesures, nous avons conclu qu'en moyenne, la probabilité que les étudiants qui ont gradué du secondaire et se sont inscrits au programme de sciences humaines en 2006 possèdent les connaissances prérequises en algèbre est de 23 % et la probabilité que ces étudiants aient un niveau de connaissances satisfaisant en fonctions est de moins de 20 %. Cependant, cette étude démontre que le fait de posséder les connaissances de base en algèbre ou en fonctions n'avait pas grandement affecté la performance des étudiants du cours de calcul différentiel. Au contraire, les connaissances en algèbre et en fonctions des étudiants avaient augmenté considérablement dans le processus d'apprentissage du calcul différentiel. Ceci pourrait probablement expliquer pourquoi les connaissances acquises auparavant par les étudiants n'avaient pas eu un impact significatif sur leurs notes finales ni sur leur apprentissage durant le cours. Il semble que les étudiants et les professeurs compensaient pour ce manque des connaissances prérequises.

2. Lesquelles de ces trois conditions expérimentales (chacune utilisant une des trois stratégies pédagogiques décrites auparavant) est la plus susceptible d'inverser la tendance de l'augmentation du taux d'échec?

Nous avons déterminé qu'il n'y a aucune différence entre l'apprentissage et la persévérance des étudiants qui soumettaient leurs devoirs et ceux qui les faisaient en ligne. Le simple ajout de la technologie dans ce processus, comparé à la méthode traditionnelle, n'aura toutefois pas contribué à inverser cette présente tendance négative. Cependant, nous devons faire remarquer que la plupart des professeurs de mathématiques du cégep ne donnent pas de feedback aux étudiants de façon hebdomadaire, en corrigeant et retournant leurs devoirs. Ceci est dû au fait que cette méthode constitue une surcharge de travail excessive pour les professeurs. Dans ce sens, l'utilisation du système de devoirs en ligne est une amélioration en comparaison avec l'ancienne pratique pédagogique.

Une autre conclusion très importante est le fait que l'apprentissage des étudiants ayant leurs devoirs à faire en ligne, combiné à l'enseignement et au soutien pour les travaux reçu en classe, surpassaient celui des étudiants soumis aux deux autres conditions. De plus, les étudiants dans ce système d'apprentissage ont mis plus d'efforts dans leurs études et étaient

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enclins à être plus persévérants. Par conséquent, cette stratégie pédagogique pourrait être utilisée pour réduire le taux d'échec si élevé dans les classes de calcul différentiel dans le programme de sciences humaines.

Nous avions aussi découvert une autre tendance plutôt inquiétante au cours de cette étude : la pratique des professeurs qui est de compenser les lacunes des étudiants en remontant leurs notes à la fin du semestre. Selon les codeurs indépendants, l'augmentation de la note moyenne était la plus élevée dans le groupe d'étudiants ayant un niveau d'apprentissage plus faible. Donc, nous pouvons affirmer que les notes finales dans ce cours ne reflètent pas avec exactitude la maîtrise des concepts du calcul différentiel démontré par les étudiants. Si ce résultat est indicatif des tendances générales des professeurs, les administrateurs d'écoles secondaires et des collèges, ainsi que les administrateurs du MELS qui attribuent la réussite des étudiants avec de telles notes, devraient être concernés. Nous recommandons que plus de recherches soient faites pour confirmer si ces pratiques sont utilisées en général et comment elles peuvent être évitées.

3. Il y a t-il des différences enter les sexes dans l'impact de ces trois conditions, dans la réussite et la persévérance des étudiants?

Dans cette cohorte d'étudiants en sciences humaines, l'apprentissage des femmes a surpassé de manière significative celles des hommes dans les trois conditions, et elles étaient aussi beaucoup plus enclines à poursuivre leurs études en mathématiques que leurs collègues masculins. Malheureusement, l'incidence de cet effet significatif était trop minime pour déterminer une différence entre les sexes dans la réussite et la persévérance dans chacune des autres conditions.

4. Il y a t-il des différences entre les sexes dans l'impact de ces trois conditions, sur la motivation et la connaissance de la propre capacité des étudiants?

Il n'y avait aucune différence dans la motivation des étudiants masculins ou féminins ou due aux trois autres conditions expérimentales. Au contraire, il y avait une importante interaction entre les conditions et les sexes en termes de connaissance de sa propre capacité. Comme étant souvent le cas dans la recherche d'éducation en mathématiques, nous avions

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déterminé que la connaissance de ses propres capacités des femmes pour les conditions C1 et C2 était moins élevé que celles des hommes. D'autre part, la connaissance de ses propres capacités des femmes dans la condition C3 était beaucoup plus élevée que celles des hommes. Avec cette revalorisation de leurs compétences en mathématiques, ces femmes seront plus enclines à persévérer, à déployer plus d'efforts et à essayer d'acquérir une solide connaissance en mathématiques pour être prêtes à surmonter les obstacles dans leur future carrière.

Nous avons identifié une stratégie pédagogique (C3) qui a permis de promouvoir avec succès la réussite et la persévérance des étudiants du cours de mathématiques. Bien que ceci ait été expérimenté chez des étudiants en sciences humaines du cégep, il n'y a rien de particulier à signaler sur cette population qui pourrait indiquer que cette stratégie pourrait échouer ni avoir un impact sur des étudiants du secondaire ou du cégep suivant des cours de mathématiques dans d'autres programmes. L'utilisation de cette stratégie dans les écoles secondaires et les cégeps pourrait inverser ce double problème de la réduction du taux d'inscriptions aux cours de calcul différentiel en sciences humaines et le taux d'échec élevé. Le Québec, avec un taux élevé d'abandon de femmes en programme de sciences au cégep, pourrait se servir de cette stratégie pédagogique, en classes de sciences pour augmenter le nombre de graduées en sciences.

Ainsi donc, un investissement modeste dans la technologie pourrait améliorer le rendement des étudiants québécois, les aidant à atteindre leur potentiel et, du même coup, augmenter la concurrence du Québec face à d'autres pays industrialisés. Cette constatation est le résultat le plus important de cette recherche.

Summary

Summary

Key words: College Education, Mathematics, Achievement, Perseverance, Self-efficacy

Over the past decade enrollment and student achievement in Calculus courses in the Social Science Program have been declining. This study aimed to determine whether these trends could be reversed by use of an improved instructional design, principally by adding a web-based homework system, WeBWorK, as a course component. To this end, three instructional designs were developed and assessed in the course of this study: traditional lectures, coupled with paper-based assignments (C1); traditional lectures coupled with WeBWorK assignments (C2); traditional lectures, coupled with in-class interactive sessions designed to provide teacher and peer support for students working on WeBWorK assignments (C3). We examined the outcomes of this quasi-experimental study, in terms of students' academic performance, persistence in mathematics courses, self-efficacy and motivation. The academic performance was assessed by final grades as well as by achievement in Calculus, with the latter obtained by having independent coders code students' work. In addition, we assessed students' prior knowledge of algebra and functions and examined whether students' prior knowledge impacts on their learning of Calculus. There were four research questions in this study:

1. Are students ill prepared to study Calculus, and if so, does their ill preparation have a significant impact on their learning in Calculus?

To answer this question, we developed and validated, both intrinsically and extrinsically, two measures of knowledge of algebra and functions. These measures may become an important tool for instructors who can use them to assess their students, and consequently plan remedial strategies to help students overcome gaps in their pre-requisite knowledge. Using these measures we determined that, on average, the probability that students who graduated from high school and who enrolled in the Social Science Program in 2006 have sufficient knowledge of algebra pertinent to Calculus is 23%, and the probability that these students have satisfactory knowledge of functions pertinent to Calculus is less than 20%. However, this research shows that prior knowledge of algebra or functions did not significantly affect students' performance in Calculus. Students knowledge of algebra and functions grew significantly during Calculus instruction. This may explain why prior knowledge did not significantly impact upon students' final grades or

achievement in Calculus. It seems that both students and instructors compensated for the gaps in initial knowledge.

2. Which of the three tested experimental conditions (instructional designs) is most likely to reverse the trend of continually increasing failure rates?

We determined that there were no differences in achievement and perseverance between students who submitted paper assignments and those who did the same assignments online. Simply adding computer technology to a traditional instructional design would not reverse the current downward trend. However, we should point out that providing feedback in the form of weekly assignments, corrected and returned, is no longer the standard practice of CEGEP Calculus instructors due to prohibitive workload demand of such a practice. In that sense, the addition of online homework assignments is an improvement over the current practice of assigning but not collecting and correcting homework.

A more important conclusion is that students who had online homework assignments, combined with instructional support in class for those assignments, outperformed students in the other two conditions. In addition, students in this learning environment put more effort into their studies and were also likely to persist more. Consequently, this instructional design could be used to alleviate the high failure rates in social science Calculus classes.

We also discovered a worrisome trend in the course of this study: one method that instructors use to compensate for initial weaknesses of students is to "boost" grades at the end of the term. The average grade "boost" was significantly higher in the group of students that had the lowest scores as assessed by independent coders. Consequently, we may say that the teacher assigned final grades in the course did not accurately reflect students' mastery of Calculus concepts. If this result is indicative of general trends in grading, then high school and college administrators, as well as MELS administrators, who monitor students' success via such grades, should be concerned. We recommend that further research into causes and possible remedies of grade boosting should have a high priority.

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3. Are there gender differences in the impact of these three conditions on student achievement and perseverance?

In this cohort of social science students, women significantly outperformed men in all three conditions and they were also significantly more likely to pursue further studies in mathematics than their male peers. Unfortunately, the effect size was too small to show gender differences in student achievement and perseverance within each of the conditions.

4. Are there gender differences in the impact of these three conditions on student motivation and self-efficacy?

There were no differences in the motivation of students due to gender or due to any one of the three experimental conditions. On the other hand, there was a significant interaction between condition and gender in terms of self-efficacy. As is often the case in mathematics education research, we determined that the self-efficacy of women in conditions C1 and C2 was lower than that of men. On the other hand, the self-efficacy of women in condition C3 was significantly higher than the self-efficacy of men. By having their beliefs about competence in mathematics enhanced, these women will be more likely to persevere, to expend more effort and to gain solid background in mathematics and to be ready to face challenges in their future careers.

We have identified a successful instructional strategy (C3) that was shown to promote achievement and perseverance of students in mathematics courses. Although, it has been tested amongst CEGEP social science students, there is nothing specific about this population that would indicate that this strategy might fail to similarly impact high school students, or CEGEP students taking mathematics in other programs. Use of this strategy in both high school and CEGEP is likely to reverse the dual problems of declining enrolment in social science Calculus classes, and the high failure rates there. Since Quebec also faces the problem of a large number of women tending to drop-out of the CEGEP Science Program, this instructional strategy in science program classes would increase the number of female science graduates.

Thus, a modest investment in technology could improve the performance of Quebec students, helping them to fulfill their potential, thereby increasing Quebec's competitiveness vis-a-vis other industrialised countries. This is the most important finding of this research effort.

Introduction

I. Introduction

In the late 1500's Clavius first introduced the subject of mathematics to university studies. Since then people have struggled with the dual problem of how to teach and how to learn mathematics (Smolarski, 2002). Complaints about mathematics instruction are not a new phenomenon. In the *Bulletin* [October 1900, pp. 14-24], from the meeting of the American Mathematical Society in 1900, we find this statement: “The fundamental principles of Calculus must be taught in a manner wholly different from that set forth in the textbooks ...” (Ewing, 1996). A hundred years later we still debate what mathematics to teach and how to teach it. Meanwhile, student success and understanding ebb. The Conseil des Ministres de l’Éducation (1997) reported that in Québec schools student achievement in mathematics declined from 1993 to 1997. Similarly, student achievement in mathematics in U.S. schools, as measured by the National Assessment of Educational Progress, declined steadily from 1970 through the early 1980's (National Centre for Education Statistics, 1997). G. Nelson, the Director of Project 2061 (2004) believes that the problem is caused by mathematics curricula that “emphasize quantity over quality, and are all a mile wide and an inch deep”.

In the twenty first century mathematics is the gateway to careers in many fields, ranging from the sciences to economics, commerce and other social sciences. Increasing use of sophisticated statistical analysis in the social sciences compels students to complete Calculus courses that precede statistics. At Vanier College, in the 2001 cohort, the last year for which data was relatively complete at the inception of this project, there was a 41% failure rate amongst the 55% of social science students who enrolled in Calculus I (computed from Profil Scolaire des Étudiants par Programme (PSEP), SRAM, 2005). This means that 68% of our social science students cannot pursue careers involving mathematics. The situation across the network of colleges associated with SRAM appears to be even worse, with a 26% failure rate but only 20% of social science students taking Calculus I, so that 85% of social science students cannot pursue careers involving mathematics. (Note that the PSEP data does not allow analysis based on marks obtained in High School mathematics courses, which might explain some of the above differences.) High failure rates in CÉGEP Calculus I courses are not unique in the North American context. Recent NCTM studies show that 50% of students fail college math courses (Gordon, 2005a). Gordon is both humorous and serious when he states “As several physicists have put it, “the half-life of math students is one semester”. In an increasingly quantitative

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society, this should not be acceptable." We note that the failure rate is higher amongst male students than amongst their female counterparts.

Current trends in the CEGEP system indicate declining enrollment of social science students in Calculus courses, illustrated at Vanier College by a drop in the percentage of social science students taking Calculus I from 74% of the 1994 cohort to 55% of the 2001 cohort. Similar trends are observed across the CEGEP network (Profil Scolaire des Étudiants par Programme, SRAM, 2005). Summary statistics from the registrar of Vanier College indicate that this problem of social science students' low enrollment in Calculus courses persists, enrollment decreasing significantly from 39.4% in 2004 to 31.2% in 2006 (Pearson χ^2 (2, 3322) = 16.516, $p < .001$). This continuing decline cannot be attributed to decreasing students' high school performance in mathematics since average grades in high school mathematics courses remained constant over that period. Alarmingly, 10.3% of students in these three cohorts, graduating from the highest level mathematics courses at both Secondary IV and V, with distinction (an average grade of 75.12), decided not to pursue CEGEP mathematics courses. Further, although women formed the majority in two of the three cohorts, in all three cohorts fewer women enrolled in mathematics courses.

This study aimed to determine whether social science students' success and perseverance in Calculus courses could be improved, reversing current trends. To this end, three instructional strategies were examined in Calculus classes. We report below on the outcomes of this experiment, in terms of students' academic performance (grades and knowledge of Calculus) and persistence in mathematics courses, and then on the implications of this research for the CEGEP network.

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II. Theoretical Perspective

Mathematicians and Educational Researchers

Two distinct groups struggle to understand the underlying causes for declining achievement in mathematics, and to develop programs that will reverse this trend. The first group can be called mathematicians, and consists of mathematics instructors and working mathematicians. The second group consists of researchers in education, and specifically in mathematics education. These two groups, with different perspectives, address issues very differently. There is a tendency amongst mathematicians to blame other mathematicians, *e.g.*, college instructors routinely blame failures in their Calculus courses on high school instructors, who, in their view, do not sufficiently train students' algebraic skills. They also tend to attribute failure rates to students' lack of motivation to study and work through all assigned problems. Educational researchers tend to take a very different view. They seriously question the wisdom of the curriculum, particularly the focus on a huge variety of technical "tricks", and the heavy emphasis on algebraic manipulation in Calculus courses, and call for curricular revision. They also examine how theories of human motivation apply to different student populations (male, female, ethnic, underprivileged, etc.) in the context of Calculus courses, and on the basis of their findings suggest different pedagogical approaches. Unfortunately, mathematicians tend to mistrust both the research results, and the very methodology of educational research (Schoenfeld, 2000; Klein, 1996). Consequently, the findings or conjectures made by educational researchers rarely impact upon the daily practice of the instructors. On the other hand, researchers tend to dismiss the concerns of mathematicians as anecdotal evidence. In this study, which targets the issue of high failure rates in Calculus classes, in particular in relation to gender, we attempt to address the concerns of CEGEP mathematics instructors while at the same time taking into account the findings of educational researchers.

Disjunction between High School and CEGEP

In response to alarming failure rates in mathematics courses, the reformers of high school mathematics courses implemented instructional designs and a curriculum that is rooted in a constructivist perspective on teaching and learning. In a constructivist perspective, when students learn, they formulate representation(s) of new concepts by relating them to concepts that already exist in their knowledge structure (*e.g.,* see an excellent summary concerning a constructivist perspective in Lafortune, 2001, p. 24). These mental representations are created by

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external stimuli, such as observations of events in the external world, listening to explanations provided by teachers or peers, physical manipulation of objects, etc. It is important to note that these representations are not fixed. As learners resolve cognitive conflicts, their representations of a concept become more complex, the network of links between that concept and other concepts in their knowledge structure grows, and consequently their knowledge structure changes. As a result, proponents of a constructivist paradigm believe that students acquire deeper conceptual understanding. This notion of learning was recently disputed by cognitive psychologists, in particular Kirschner, Sweller and Clark (2006), who believe that direct instruction produces better results than constructivist instructional designs.

In addition, reformers of the Québec high school curriculum have indicated a belief that the traditional heavy emphasis on algebraic manipulation in mathematics courses drowns students in a sea of calculations, preventing the essential formulation of a multitude of representations of concepts. The new high school curriculum exposes students to a balance of graphical, numerical, verbal and algebraic representations of functions, an effort similar to the Calculus reform movement dating back to the eighties (Hodgson, 1987). It allows students to learn concepts in their own preferred mode of representation. Furthermore, it encourages students to emulate the behaviour of expert mathematicians, who move fluidly between all four representations to gain additional insights. However, research (*e.g.,* a metanalysis of 52 studies by Barton, 1996) studying the impact of using multiple representations on students' achievement is inconclusive. There are studies that show that students benefit from using multiple representation (*e.g.*, Ainsworth, Wood & O'Malley, 1998; Cox & Brna, 1995; Dedic, Rosenfield, Alalouf & Klasa, 2004). Dedic *et al.,* (2004) found that using multiple representations had a positive impact on student performance, particularly on tasks related to the graphing of functions. Other researchers have shown that students have difficulties working with multiple representations and doubt the positive impact on students' achievement (*e.g.*, Tabachnick, Leonardo & Simon, 1994; Ainsworth, Bibby & Wood, 1997; Klein & Rosen, 1996).

Reformers of the Québec high school curriculum also focussed more on realistic applications through mathematical modelling. When mathematical models are applied to realistic and interesting problems, students create richer mental representations. To facilitate the use of graphical and numerical representations, as well as to focus on modelling, high school students are now expected to be trained to use graphing calculators in their mathematics classes. Thus,

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current high school graduates have skills that are seldom used in typical Calculus I classes (Gordon, 2005b), where many instructors ban the use of graphing calculators altogether. At the same time, the emphasis in high school has shifted away from algebraic skills and thus, it is probably true that most students who currently graduate from high schools in Québec, having passed either Mathematics 526 or Mathematics 536, are less versed in algebraic skills than most graduates from comparable courses in the past. However, Calculus I instructors focus on tasks that require a high level of algebraic skills, and most of our current students just do not have that level of skill. It appears to us that the impact of this mismatch in curriculum needs to be addressed if we are to truly assure improved students success. This is particularly the case for social science students, who may feel that the heavy emphasis on algebraic manipulation found in traditional Calculus courses is a daunting obstacle, perhaps explaining both the reluctance to enroll and the subsequent large failure rates. Some researchers in mathematics education (Gordon, 2005a) propose a reduction in the content of Calculus courses. If done wisely, this may not be such a bad idea, given the comment made by Peter Lax, Past President of the American Mathematical society, who said "Calculus as currently taught is, alas, full of inert material ..." (Uhl, 2000), but this point of view is opposed by many mathematics instructors.

The instructional design that we tested in this study should help to overcome some consequences of the disjunction between the high school and CEGEP curriculum. The design included a computer assisted instruction component, with a heavy emphasis on both conceptual understanding and transitions between multiple representations. In this manner, students had an increased opportunity to use skills learned in high school. Further, computers were used to provide enough algebraic exercises to strengthen students' skills, but at no loss of class time nor increase in correction time for college instructors.

Self-efficacy & Motivation: A Research Perspective

College instructors also claim that students are not motivated to study and do not do homework problems. They may be right, but if so, the useful questions to pose are: why is it so; and, what can teachers do to motivate their students? Many educational researchers are currently searching for answers to the latter question. Social cognitive theory, a popular perspective on human motivation, has emerged in recent years from the work of several theorists (Bandura, 1986; Bandura, 1997; Bandura & Locke, 2003; Pajares, 1996; Pajares & Schunk, 2001; Zimmerman & Martinez-Pons, 1990; Zimmerman, Bandura & Martinez-Pons, 1992). In this

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theory people are seen as self-organizing, proactive and self-regulating, rather than reactive and governed by external events. Self-efficacy, a personal judgement of one's capability to accomplish a task, stands out as having the greatest direct mediating effect on human psycho-social functioning. Self-efficacy beliefs are domain specific: one may believe oneself to be capable of correctly solving Calculus problems while simultaneously believing oneself incapable of writing a good essay, or vice versa. Academic self-efficacy judgements have been shown to correlate positively with academic performance in mathematics (Pajares & Miller, 1995) and persistence in academic tasks (Pajares, 2002). In addition, according to Self-determination theory (Deci & Ryan, 2000) learning environments that promote student autonomy positively impact on student motivation to strive for higher achievement. When students feel autonomous, competent and related, their self-efficacy rises and consequently they tend to persevere in their studies. This was certainly observed amongst CEGEP science students (Dedic, *et al.*, 2007).

Academic Self-efficacy & Gender

In the context of this research it is important to note that the self-efficacy beliefs of female students and male students differ (Pajares, 2002). Women have greater confidence than their male counterparts in their use of self-regulated strategies (*e.g.,* completing homework on time, general time management, etc.). On the other hand, women tend to have lower self-efficacy beliefs concerning mathematics.

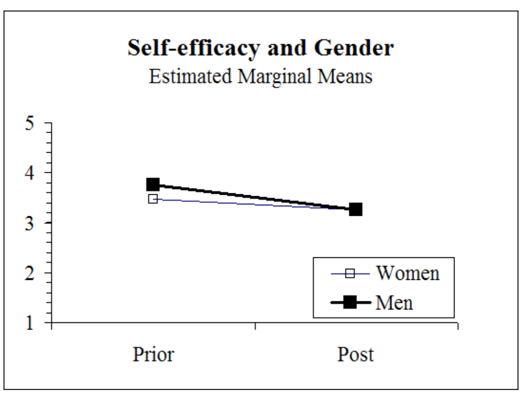

Figure 1

In our study of 2500 newly enrolled students in public Anglophone CÉGEPs in 2003 (Rosenfield, Dedic, Dickie, Aulls, Rosenfield, Koestner, Krishtalka, Milkman, & Abrami, 2005), we found that male students had significantly ($p < 0.001$) higher self-efficacy beliefs in mathematics than women at the time of enrollment ("Prior" in the graph on the left). After one semester of Calculus instruction the self-efficacy beliefs (in mathematics) of both men and women significantly decreased, but the decline amongst male students was higher (all changes of self-efficacy in mathematics were significant at $p < 0.001$). These results confirm that self-efficacy beliefs in mathematics are not stable and vary

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with academic experience. The negative impact of low self-efficacy beliefs in mathematics amongst women in social science Calculus classes may be compensated for by their high use of self-regulation strategies. However, a drop in male self-efficacy beliefs concerning mathematics, when males already use self-regulation strategies infrequently, may result in the high failure rates observed amongst male students in social science Calculus classes. Thus, to improve academic success, the problem becomes one of determining how to modify instruction in such a way as to stop, or perhaps even reverse, the decline of mathematical self-efficacy beliefs. In addition, it is important to study whether gender differences in self-efficacy beliefs, as well as gender differences in achievement and perseverance, persist in modified learning environments.

Feedback and Academic Self-efficacy

Academic self-efficacy beliefs are formulated through interactions with learning environments. For example, Siegle (2003) demonstrated that student self-efficacy increased in experimental mathematics classes where teachers provided elaborative feedback focussed on promoting self-efficacy. As students age, their self-efficacy judgements become more predictive of academic outcomes because they become more skilled at interpreting information from their interactions with learning environments (Phan & Walker, 2000). There are four sources of information: mastery experiences ("success breeds success"); vicarious experiences (observation of other people succeeding at a task); social persuasion (verbal comments made by teachers or peers); and, affect (experiencing positive or negative emotions in school). Men tend to change self-efficacy beliefs primarily in response to mastery experiences, while women are more likely to change their self-efficacy beliefs in response to vicarious experiences and social persuasion (Zeldin & Pajares, 2000). Negative verbal feedback is much more powerful in lowering self-efficacy beliefs than positive verbal feedback is in raising self-efficacy beliefs (Schunk & Pajares, 2002). The important conclusion to draw from the literature is that students get feedback from the classroom learning environment, which affects their self-efficacy beliefs. In turn, those self-efficacy beliefs impact on their motivation to succeed.

Feedback, Homework and Motivation in CEGEP Mathematics

Currently, in a typical CEGEP mathematics learning environment, the teacher presents a new concept, and then assigns problems that students can only solve if they have understood the concept. Although most CEGEP instructors assign weekly homework, because of their workload they rarely collect/correct homework. That is, teachers ask students, largely on their own, to do

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problems, monitor success, and self-correct understanding, all to be done until new concepts are mastered. From the perspective of socio-cognitive theory (Bandura, 1997), this type of internal feedback loop works well only for highly self-efficacious students possessing appropriate self-regulatory strategies (Zimmermann and Pons-Martinez, 1990). It is unlikely that such educational practices promote effective learning for any other group of students. When ineffective learning processes are followed by summative assessment, the combination delivers an educational one-two punch, diminishing self-efficacy beliefs and effort expended in completing assignments, all of which further lowers achievement. However, many instructors observing poor student performance draw a different conclusion, namely that, lacking the incentive of marks, students are not motivated, and just won't do homework. High failure rates result and neither teachers nor students see how to change the situation. The key component, missing in this sadly too common scenario, is effective feedback to/from students from/to teachers during learning (Crouch and Mazur, 2001; Buttler and Winne, 1995). Unfortunately, college instructors have a workload that prohibits weekly homework correction, and do not have teaching assistants to do such corrections.

Alarming data showing declines in the number of science, mathematics, engineering and technology (SMET) graduates, and research showing that teaching is largely to blame (*e.g.,* Seymour and Hewitt, 1997), have prompted a flurry of research efforts directed at improving mathematics instruction. The importance of homework as one of the crucial elements of improved instruction has been established by many meta-analytical studies (*e.g.,* Warton, 2001). In this context, and in view of the fact that the Web has become easily accessible, many Web based homework systems were developed over the last two decades (*e.g.,* CAPA in physics, WeBWorK in mathematics), and instructional designs which incorporate such systems have become one of many modalities of what is broadly referred to as computer aided instruction (CAI). Research into the impact of using web based homework systems on student achievement is growing (*e.g.,* Bonham, Beichner & Deardorff, 2001) but the results are as yet inconclusive. In particular, many studies report gender differences in achievement and motivation in CAI (*e.g.,* Butler, 2001). While men tend to thrive in CAI classrooms, women tend to prefer human interaction over working with computers. In the context of a web based homework system, we anticipated that men would benefit more because these systems may compensate for their lower self-regulatory skills.

Theoretical Perspective

The award winning online assessment system, WeBWorK, has been adopted by a large number of institutions in USA and Canada and the impact of the use of this system on students has been studied (*e.g.,* Gage, Pizer & Roth, 2002; Hirsch & Weibel, 2003; Hauk & Segalla, 2004; Segalla & Safer, 2006). These studies determined that using WeBWorK to deliver homework problems significantly improved the academic achievement of those students who in the end actually did the homework. Thus, usage of this system might address key issues such as the inadequacy of teacher feedback to students and student feedback to teachers, while providing the possibility for copious amounts of practice for students without requiring the huge additional expense of hiring markers. In addition, Weibel and Hirsch (2002) reported student comments that WeBWorK's instant feedback helped them to monitor their own learning progress. Although those comments were not systematically collected in their study, such comments suggest that use of this system may positively impact on student motivation.

Providing Feedback with WeBWorK

WeBWorK has many features that make it useful for mathematics educators:

- students may access problem sets from any computer with an Internet connection, and they are provided with instantaneous feedback (correct/incorrect answer);
- the system can be used to deliver assignments, quizzes, exams, diagnostic tests, or be a tool for teaching in class;
- students can work together on problem sets, but cannot simply copy solutions because each student is assigned problems with randomized parameters;
- instructors set limits on the number of tries allowed;
- instructors set the due date for each assignment (which can be altered for the whole class or for individual students, even while students are working on it);
- statistical data concerning progress of individual students (*e.g*., history of attempts for each problem) and of the whole class are automatically generated by WeBWorK and available in real time for the instructor (allowing for "just in time teaching" where the instructor can use information generated by WeBWorK to focus his instruction);
- evaluation routines allow for problems where the expected answers are: numbers, functions, symbolic expressions, arrays of yes/no statements, multiple choice questions;
- while students use calculator syntax to enter symbolic expressions, a preview screen allows them to see the expression in typeset mathematical notation;
- a large collection of ready-to-use problem sets for many mathematics courses is available in

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the WeBWorK database (problem sets were assembled by a large number of mathematical educators and tested on thousands of students, and new problems are constantly generated, discussed and shared within the WeBWorK user community);

- instructors using the system can modify existing problems, write new ones patterned on existing ones, and with programming expertise, add their own answer evaluator routines.
- allows instructors to assign a large number of practice problems without the heavy grading burden otherwise required to generate constant feedback to students.

III. Objectives

The objective of this quasi-experimental (Campbell & Stanley, 1963) study was to develop and test an instructional design for the Social Science Program Calculus course which could:

- increase students' achievement, and thereby decrease failure rates in Calculus classes;
- increase student motivation;
- enhance students' perseverance;
- compensate for the lack of algebra skills necessary for success in Calculus.

Three instructional designs were assessed in this study: traditional lectures coupled with paper-based assignments (*Condition* C1); traditional lectures coupled with WeBWorK assignments (*Condition* C2); traditional lectures coupled with WeBWorK assignments and in-class interactive sessions designed to provide both teacher and peer support for students working on WeBWorK assignments (*Condition* C3).

IV. Research Questions

1. Are students ill prepared to study Calculus, and if so, does their lack of preparation have a significant impact on their learning in Calculus?
2. Which, if any, of the three tested experimental conditions (instructional designs) is most likely to reverse the trend of increasing failure rates?
3. Are there gender differences in the impact of these three conditions on student achievement and perseverance?
4. Are there gender differences in the impact of these three conditions on student motivation and self-efficacy?

In view of the studies by Weibel and Hirsch (2002) and Gage, Pizer and Roth (2002), we hypothesized that an implementation of WeBWorK, combined with in-class interactive sessions (similar to “interactive engagement” as defined by Hake (1998)), would promote students’ success and perseverance in Calculus just as “interactive engagement” does in physics (Hake, 1998). We also hypothesized that men would be more likely to benefit from using a computer assessment tool such as WeBWorK, and that WeBWorK usage would lead to decreased male failure rates and increased achievement. Finally we hypothesized that students’ motivation and self-efficacy would be higher in classes that used WeBWorK.

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V. Methodology

Participants were social science students who enrolled in the Calculus I course at Vanier College in the fall term of 2006. There were 354 students (42.1% women and 57.9% men) who agreed to participate. Eight instructors, teaching nine intact classes of Calculus I, also agreed to participate. The nine classes were assigned to three experimental conditions, three classes each, on the basis of instructors' preference for the instructional design to be used in each condition. Thus, 118 (38.1% women, 61.9% men) student participants were enrolled in experimental *Condition* C1; 114 (38.6% women, 61.4% men) students were enrolled in *Condition* C2; and 122 (49.2% women, 50.8% men) students were enrolled in *Condition* C3. Students knew nothing of the differences in sections prior to the first week of classes, so "section enrolled in" could not have been influenced by the three conditions.

Procedure

Participating instructors met with researchers before the course began and agreed to a common textbook and a set of ten common problem assignments (see Appendix E). Hoping to increase social science students' motivation to study mathematics by increasing its relevancy, the instructors agreed to use social science applications more frequently than in past years. Thus, most assigned problems refer to situations encountered in either business or sociology. The instructors also agreed to give three term tests containing some common questions (see Appendix B), a comprehensive common final examination (see Appendix C), and to use a common evaluation schema.

The three instructors in *Condition* C1 lectured in class and assigned paper versions of problem sets. Corrected assignments were returned to students one week after submission. The two instructors in *Condition* C2 also lectured in class, but assignments were WeBWorK based, with an unlimited number of tries. Students in *Condition* C2 obtained instantaneous feedback (correct/incorrect) from WeBWorK and were encouraged to try again when their solution was incorrect or to seek help from peers or teachers. *Condition* C3 differed from *Condition* C2 solely in that the three instructors engaged students to work on WeBWorK based problems for approximately one hour per week (20% of class time) in a computer lab. During these in-class interactive sessions students were encouraged to seek help from the instructor or from their peers, while working either alone or in groups with computers.

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Variables:

Independent Variables:

Condition and *Gender*

Dependent Variables and Covariates:

Achievement (*High School Math Performance*; *Knowledge of Algebra*; *Knowledge of Functions; fscore; fgrade*);

Perseverance (*Perseverance, Probability*);

Effort (*Assignment*, *Frequency*);

Motivation (*Amotivation, Self-determined Motivation, Extrinsically Determined Motivation*);

Self-efficacy; Perception of Learning Environment.

Measures: Achievement

Student high school performance was assessed using grades from mathematics courses taken in Secondary IV and Secondary V. Québec high school students choose one of three different levels of mathematics courses. Consultations with expert high school teachers revealed that the content of the lowest level courses, 416 and/or 514, is substantially reduced in comparison to the higher level courses. The content of the second level courses, 426 and/or 526, is essentially the same as the content of the highest level courses, 436 and/or 536, the difference lying primarily in the difficulty of the problems students are expected to solve. To account for these different levels, we used an algorithm developed in previous research (Rosenfield *et al.*, 2005), reducing grades obtained in the lowest level course by a factor of 0.7, and increasing grades obtained in the highest level course by a factor of 1.1. In this manner the scale of student performance is stretched, ranging from 0 to 110. Then, a variable, *High School Math Performance*, was computed as the average performance in Secondary IV and Secondary V. This variable assesses student prior achievement.

Students' academic performance at the CEGEP level was assessed by their final grade (*fgrade*) in the Calculus course. Students' knowledge of Calculus was also assessed independently from instructors' grading practices. Over the course of the semester instructors included a set of 13 problems in the three term tests, and students' solutions were photocopied by the researchers prior to teacher correction. In addition, researchers photocopied students' solutions to the common final exam, 10 problems, again prior to teacher correction. (See

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appendices B and C for the common term test problems and the final exam respectively.) Coding schemas were developed for all 23 problems (see appendix D), and two independent coders coded all student solutions. The inter-coder reliability was assessed to be in excess of 92%. Grades for each student were then computed based on the coding using scoring schema (see appendix D). In addition, all students were given ten homework assignments (see Appendix E), which were scored (percentage of correct answers) either by WeBWorK (*Condition* C2 or *Condition* C3) or by an independent coder (*Condition* C1). A common evaluation schema (20% assignment grade and 80% coded term test problems and final examination) was used to compute a variable (*fscore*) that provides an assessment of students' knowledge of Calculus independent of instructors' grading. In addition, we computed the percentage of correctly solved problems on assignments (*Assignment*), and the frequency of submission of assignments (*Frequency*) as measures of student effort in the course. Perseverance was assessed using students' academic records in a variable (*Perseverance*) where 1 indicates that a student took only Calculus I, and 2 indicates that a student enrolled in Calculus II the next semester. Students may also enrol in Calculus II and/or Linear Algebra in their third or fourth semester of collegial studies. To improve our assessment of perseverance by accounting for the possibility of taking math courses later, we computed the probability of perseverance (*Probability*) in mathematics. Logistic regression was performed with *Perseverance* as outcome variable and two continuous predictors (*fgrade* and *High School Math Performance*). Results indicated that the full model against constant-only model was statistically reliable χ^2 (2, 318) = 168.146, $p < .001$ with Nagelkerke R^2 equal to .548. The classification table reveals that the model satisfactorily classifies participants since it correctly predicts 77.8% of non-persisters and 80.6% of persisters. The probability of classification was saved as the variable *Probability* and used in subsequent analysis as a measure of perseverance.

Measures: Motivation and Self-efficacy

Student motivation was assessed using an adapted AMS survey (Vallerand, Pelletier, Blais, Brière, Senécal & Vailieres, 1992; Vallerand, 1992). This twenty-item instrument assesses the reasons for which students decide to study Calculus. It has five sub-scales: intrinsic motivation (*e.g., I study Calculus because I get pleasure from learning new things in Calculus.*); identified motivation (*e.g., I study Calculus because I think that knowledge of Calculus will help me in my chosen career.*); introjected motivation (*e.g., I study Calculus to prove to myself that I am capable of passing a Calculus course.*); externally regulated motivation (*e.g., I study Calculus*

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because without Calculus it would be harder to get into university programs that lead to high paying jobs.); and amotivation (*Honestly, I really feel that I am wasting my time in Calculus course.*). This instrument reliably assesses intrinsic motivation (α-Cronbach = .886), identified motivation (α-Cronbach = .732); introjected motivation (α-Cronbach = .761); externally regulated motivation (α-Cronbach = .746); and amotivation (α-Cronbach = .877). We also computed *Self-determined Motivation* as a construct reflecting both intrinsic and identified motivation. This scale consists of four items assessing intrinsic motivation and four items assessing identified motivation (α-Cronbach = .823). Similarly, we computed *Extrinsically Determined Motivation* as a construct reflecting both, externally regulated and introjected motivation. This scale also consists of eight items with an acceptable reliability (α-Cronbach = .786).

A six-item instrument that was used to assess students' self-efficacy in mathematics, was adapted from the MSLQ (Pintrich, Smith, Garcia, & McKeachie, 1991). Items specifically refer to students' beliefs about their competence in Calculus (*e.g., I am confident that I will be able to correctly solve problems in Calculus.*). This instrument has high internal consistency (α-Cronbach=.83) and external validity (Dedic, Rosenfield, Alalouf & Klasa, 2004). In addition, we have used a nine item instrument that was originally adapted from the Perceptions of Science Class Questionnaire (Kardash & Wallace, 2001). This instrument assesses students' perceptions of autonomy-supportive learning environment (*e.g., The teacher tried to ensure that students felt confident and competent in the course.*). This instrument also has high internal consistency (α-Cronbach=.89). In our previous work we have shown that students' perceptions of autonomy supportive learning environment positively impacts on self-efficacy (Dedic, Simon, Rosenfield, Rosenfield & Ivanov, 2007).

Measures: Development of Instruments: Knowledge of Algebra and Functions

Prior knowledge of students participating in this study was assessed using two instruments specifically developed for this task. Previous teaching and research experience indicated that the relationship between incoming high-school marks and student factual knowledge is too dependant upon the high-school attended to allow high-school marks to be a useful predictor of student success. Thus, we determined to locate a ready-to-use uniform scale that would measure the pre-instruction mathematical knowledge of the students that was relevant to their subsequent success in Calculus classes at Vanier. An Internet search for Calculus placement tests, either

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from academic institutions or provinces/states/countries, produced only tests which were too advanced, too elementary, or most frequently, tests which emphasized topics not directly pertinent for Calculus, such as Geometry or Statistics. Given no previously validated Calculus pretest for the CEGEP context, we adapted several items from the University of California at Berkeley Calculus placement test, and wrote other items locally, and then ran a pilot test of the new instrument.

A decision was made to split the pretest into two parts, one assessing student prior knowledge in Algebra, and the second testing student prior knowledge of Functions. The pilot version of the Algebra pretest had 30 items in 10 competency areas, 3 items per area. The items were a mixture of multiple choice and open ended questions. The pilot version of the Algebra test was first administered in Fall 2005 to all Social Students at Vanier taking Calculus classes. The resulting data was fitted to a 3 parameter (difficulty, discrimination, guessing) Item Response Theory model. Items deemed too difficult or too easy for this population, as well as items which failed to discriminate or which induced too much guessing, were rewritten or replaced. The revised version was run in the Winter 2006 semester on the analogous population, and the Item Response Theory analysis was repeated. Based on the resulting Item Characteristic Curves, we selected 10 items, one from each Algebra competency area, for the final version of the Knowledge of Algebra instrument. The same process was used to develop the Knowledge of Functions instrument, except that the first pilot version was deployed in the Winter 2006 semester and the final version was selected after one run of the pilot. The Algebra and Functions knowledge instruments are attached (see Appendix A).

In the first week of the Fall 2006 semester both instruments were administered to all students participating in this study as pretests so as to assess students' prior knowledge. In the last week of the same semester the same instruments were also used as post-tests of the same population. Although the Algebra and the Functions tests each had ten items probing ten slightly different skills, the data collected showed strong correlations within each test between various items. As a result we set out to discover the dimensional structure of both tests, *i.e.*, determine a small number of latent skills which generate the observed associations between tests items. Since items in both tests are either multiple choice, true-false, or graded as true-false, the observed variables are discrete. An appropriate technique for studying associations between discrete observed variables is Latent Class Analysis (Vermunt & Magidson, 2004). The Latent Class approach to factor analysis (LCFA) hypothesizes that associations between observed discrete variables are caused by a small number of latent variables (the factors) which are also discrete and ordinal. If $y = \{y_j\}$, $j = 1, \ldots, n$ is the vector of observed variables (n items) and $\theta = \{\theta_j\}$, $i = 1, \ldots, F$, is the vector of F latent variables, then the latent model describes the joint probability for a pattern of answers and latent scores as: $P(y,\theta) = P(\theta)\,[\,\Pi\, P(y_j|\theta)]$. $P(\theta)$ is a joint probability distribution of the latent factors in the population. Thus $P(\theta)$ gives the probability that a test taker is at specific levels of the latent factors, *i.e.*, this test taker possesses or doesn't possess the latent skills necessary to succeed on the test. The distribution $P(\theta)$ characterizes the population. $P(y_j|\theta)$ are the conditional probabilities for selecting a choice y_j on test item j given the distributions of latent skills. The conditional probabilities, $P(y_j|\theta)$, are the parameters of the model which characterize the test. $P(y,\theta)$ is the joint distribution of item response patterns and latent skills. The primary model assumption concerns local independence: observed variables are independent of each other when levels of the latent variables are fixed. Equivalently, the latent variables explain all associations amongst the observed variables. Since we administered the test both pre- and post-instruction, the timing of the administration of the tests was taken into account by including a covariate variable $z = \{pre, post\}$ in the models. In the presence of a covariate, all quantities describing the population, but not the test, depend upon z. Thus, the Latent Class model becomes $P(y,\theta,z) = P(\theta,z)\,[\Pi\, P(y_j|\theta)]$ and here our assumption is that the measurement properties of the Algebra and Functions test are the same pre and post Calculus instruction, with only the distributions of the latent variables (knowledge) being affected by the intervening Calculus instruction.

Models were fit to the data by maximization of the likelihood function. We used LatentGold, (Vermunt and Magidson, 2005) a commercial software package, to analyze the data. For both tests we have $n = 10$ items, and models were examined for fit by starting with one latent factor and gradually increasing the number of latent factors. Since the data tables under consideration were sparse we used a bootstrapped p-value for the likelihood-ratio chi-squared statistic L^2, and BIC and AIC information criteria to select a converging model which fit the data most parsimoniously.

Our analysis of the Algebra pre-test data showed that more than 20% of students guessed randomly on question 7, and therefore this question was modified for the post-test, and excluded from any further analysis. Analyzing data for the remaining 9 questions on the Algebra test, the accepted model has two underlying dichotomous latent variables (factors). All 9 questions had significant loadings on the first factor A1, with questions 1 and 10 loading strongest. This algebra factor, A1, will therefore be referred to as the *Knowledge of Algebra* factor. Only questions 6 and 9 had significant loadings on the second factor from the Algebra test, A2. Since both these questions deal with quadratic equations, the second factor assesses the *Knowledge of Quadratic Equations*. Since these items also load on the first factor (the two factors are not orthogonal), we anticipate problems with collinearity and therefore, we used only the first factor in further analysis.

Three questions, 4, 6 and 10, from the Functions test were changed from the pre- to the post-test and thus were excluded from further analysis. Question 8 was answered correctly by a very small percentage of the students, and thus it was deemed to be too difficult and not discriminating for this population, and was consequently also excluded from further analysis. Using data concerning the remaining 6 questions, the best fitting model we obtained had three underlying dichotomous latent factors. All six questions had significant loadings on the first factor, F1, with questions 2, 3 and 9 loading strongest. Therefore, this functions factor F1 will henceforth be referred to as the *Knowledge of Functions* factor. Primarily those questions dealing with the algebra and composition of functions loaded on the second factor, F2. Those questions testing the definition of a function loaded on the third factor, F3. However, for both F2 and F3, the classification errors of assignment at different levels of these factors were very high and therefore analysis of student scores on these latent factors was not pursued further.

Given a model which fits the data, the conditional probabilities, $P(y_j|\theta)$, of this model, together with the observed response pattern, $\{y_j\}$, for a given student, Bayes' Theorem could be used to calculate the posterior probabilities for this student being assigned at specific levels of the latent factors, $P(\theta|y,z) = [P(\theta|z)\ P(y|\theta,z)]/P(y|z)$. Since all latent factors are dichotomous, the posterior probability for assignment to the higher level on any factor serves as a convenient factor score. The score of a specific student on a specific factor can then be interpreted as the probability that she/he possesses the skill corresponding to the latent factor under consideration.

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Statistical Analysis

Given that most of the research questions are related to impact of three different treatment conditions on a set of dependent variables, multivariate analysis of variance is the most appropriate statistical method to use in this study. Since the participants were not randomly assigned to conditions, we used MANCOVA to adjust for one or more covariate variables. In each of the analyses, there were two independent factors: *Condition* (three levels: C1, C2 and C3) and *Gender* (2 levels: women and men). As we report the results of the analysis, we will always show a table of multivariate tests which show the significance of various effects and also the strength of association (Partial η^2). Wilk's criterion will be used in testing for significance in all subsequent analyses. We will also present plots of estimated partial means of dependent variables, when the effect was significant, to further illustrate the relationships. In addition, we will also report pair-wise comparisons. The latter is particularly important because one of the independent factors has three levels and we are interested in knowing the differences between individual levels. It should also be noted that the data set was tested for univariate/multivariate outliers, normality, skewness, kurtosis and multicollinearity (Tabachnik and Fidel, 2001). Twenty-two outliers were removed from further statistical analyses.

Results

VI. Results

In this section we will present:

- results concerning evaluation of prior differences in students' achievement, knowledge and motivation between the three conditions;
- results concerning students' perceptions of the learning environments in the classes participating in this study;
- results concerning the learning of algebra and functions in the Calculus classes participating in this study;
- results concerning changes in students' motivation;
- main results: differences in student achievement and perseverance due to the experimental conditions, as well as any gender differences; and
- results concerning the impact of the three experimental conditions on students' grades and effort in the course.

1. *Equivalence of Groups*

Each of the three experimental conditions in this quasi-experimental study (Campbell & Stanley, 1963) consisted of a group of three intact classes. Students were not randomly assigned to classes, nor were the classes randomly assigned to any of the three experimental conditions. It was therefore necessary to examine whether the groups of students in each experimental condition were equivalent pre-instruction, at least in terms of those characteristics that we measure because they have a large impact on the outcome variables.

<u>High-School Performance and Prior Knowledge</u>

Pre-instruction scores, which are probabilities of having adequate knowledge, averaged for all students inside each of the three conditions, on the variables *Knowledge of Algebra* and *Knowledge of Functions* are summarized in the table below:

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Table 1. Means of prior knowledge of algebra and functions.

	Condition	N	Mean	SD
Prior *Knowledge of Algebra*	C1	116	0.23	0.31
	C2	109	0.23	0.31
	C3	120	0.24	0.32
Prior *Knowledge of Functions*	C1	118	0.19	0.33
	C2	114	0.04	0.11
	C3	122	0.16	0.32

We determined that students' prior skills in algebra were low across the three conditions. The results indicate that on average the probability of successfully solving all problems on algebra test was 23 to 24%. Alarmingly, we also determined that students prior knowledge of functions was even lower. On average, the probability that students can correctly answer all questions was less than 20% in *Condition* C1 and C3, and in *Condition* C2, only 4% .

To assess the equivalence of groups in terms of prior knowledge, MANOVA, with dependent variables (*High School Math Performance*, *Knowledge of Algebra* and *Knowledge of Functions*), and independent variables (*Condition* (C1, C2 and C3) and *Gender* (Women and Men)) was carried out. Table 2. below shows the results of the multivariate tests:

Table 2. Multivariate tests: prior knowledge (N=285)

Effect	Hypothesis df	F	Sig.	Partial η^2
Intercept	3	6467.871	.000	.986
Condition	6	4.674	.000	.048
Gender	3	1.355	.257	.014
*Condition * Gender*	6	.685	.662	.007

The results above indicate a significant impact of *Condition* on a linear combination of the three dependent variables: *High School Math Performance*, *Knowledge of Algebra* and *Knowledge of Functions*. *Gender*, or interaction between *Condition* and *Gender*, do not have a significant impact. Tests between subjects indicate that there are no significant differences between the means of *High School Math Performance* in three conditions ($F(2,279)=1.473$, $p=.231$, Partial $\eta^2=.010$) or between the means of *Knowledge of Algebra* ($F(2,279)=.565$, $p=.569$, Partial $\eta^2=.004$). On the other hand, there are significant differences between the means of *Knowledge of Functions* ($F(2,279)=8.953$, $p<.001$, Partial $\eta^2=.060$), although the strength of the association is small, explaining only 6% of variance. To further investigate these differences, we

Results

computed the differences between marginal means. We determined that there are no significant differences (p=.101) between the means of *Knowledge of Functions* between *Condition* C1 and *Condition* C3. On the other hand, the mean of *Knowledge of Functions* is significantly lower (at the level .05) in *Condition* C2, in contrast with *Condition* C3 (Mean Difference (C2-C3)= -.101), and in contrast with *Condition* C1 (Mean Difference (C2-C1)=-.166).

Figure 2 at the right shows the estimated marginal means of *Knowledge of Functions*. It should be noted that scores represent the probability that students understand the concept of functions. The graph indicates that on average students have between 10% to 25% probability of understanding this concept in *Conditions* C1 and C3 and below 10% in *Condition* C2.

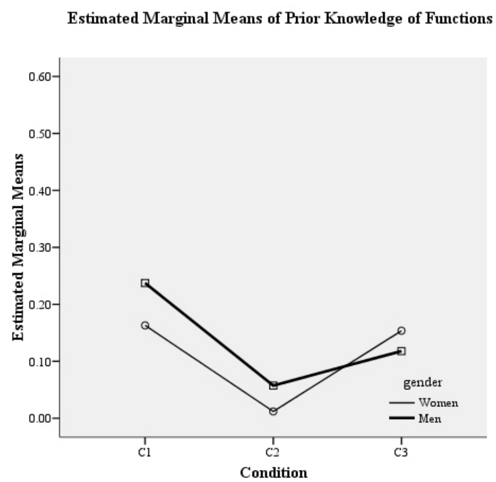

Figure 2

Prior Motivation and Prior Self-efficacy

We tested for equivalence across groups in the variables Prior *Self-efficacy*, Prior *Self-determined Motivation*, Prior *Extrinsically Determined Motivation* and Prior *Amotivation*, with *Condition* (C1, C2 and C3) and *Gender* (Women and Men) acting as independent variables. Table 3. below shows the results of multivariate tests:

Table 3. Multivariate Tests: Prior *Motivation* and *Self-efficacy* (N=311)

Effect	F	Hypothesis df	Sig.	Partial η^2
Intercept	9846.849	4	.000	.992
Condition	1.960	8	.049	.025
Gender	6.013	4	.000	.073
*Condition * Gender*	.485	8	.867	.006

The results indicate significant impact of *Condition* and *Gender* on a linear combination of the four variables: Prior *Self-efficacy*, Prior *Self-determined Motivation* and Prior *Extrinsically Determined Motivation*, as well as Prior *Amotivation.* The interaction between *Condition* and

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Gender does not have significant impact. Tests between subjects indicate that there are no significant differences between the means of Prior *Self-efficacy* across the three conditions ($F(2,310)=1.595$, $p=.205$, Partial $\eta^2=.010$) or between the means of Prior *Extrinsically Determined Motivation* ($F(2,310)=.554$, $p=.575$, Partial $\eta^2=.004$). On the other hand, there are small but significant differences between the means of *Self-determined Motivation* across the three conditions ($F(2,310)=4.286$, $p=.015$, Partial $\eta^2=.027$), and between the means of Prior *Amotivation* ($F(2,310)=6.525$, $p=.002$, Partial $\eta^2=.040$).

To further investigate these differences, we computed differences between the marginal means for the three conditions and determined that there are no significant differences in the means of any motivational variables between *Condition* C1 and *Condition* C3. On the other hand, the mean of Prior *Amotivation* is significantly higher (at the level .05) in *Condition* C2 in contrast with *Condition* C3 (Mean Difference (C2-C3)= .238) and also, in contrast with *Condition* C1 (Mean Difference (C2-C1)= .359). The graph of the estimated means of Prior *Amotivation* is shown in Figure 3 below.

Further, the mean of Prior *Self-determined Motivation* is significantly lower (at the level .05) on *Condition* C2 in contrast with *Condition* C1 (Mean Difference (C2-C1)= -.258). The graph of estimated means of Prior *Self-determined Motivation* is shown in Figure 4 below.

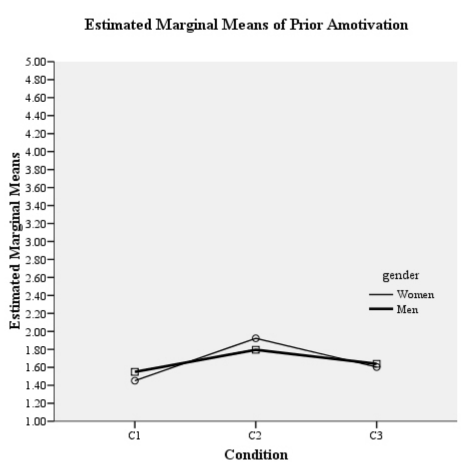

Figure 3

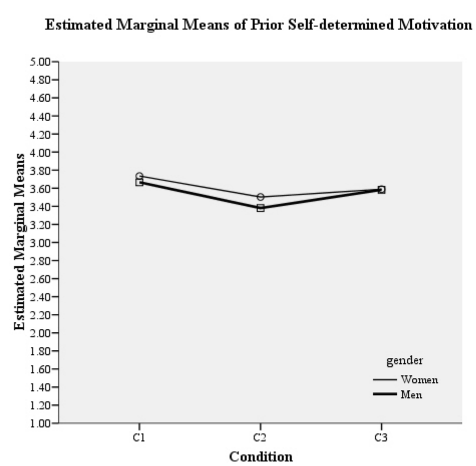

Figure 4

Results

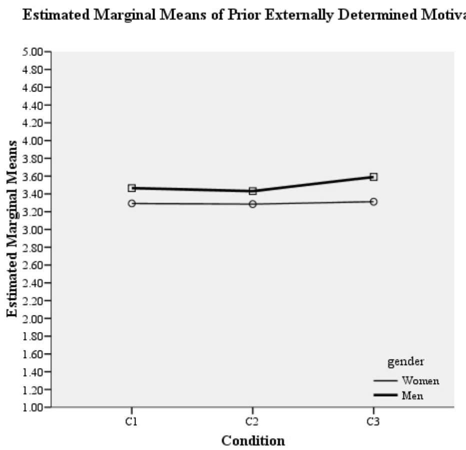

Figure 5

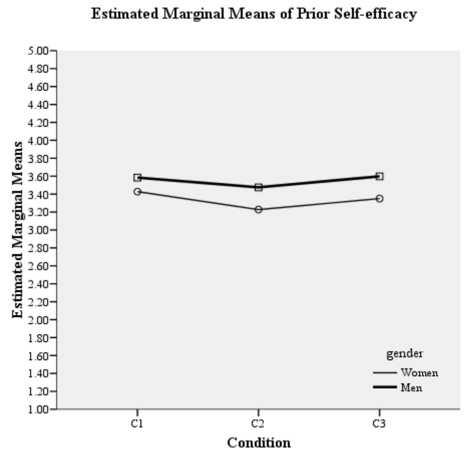

Figure 6

In addition, tests between subjects indicate that women had significantly lower Prior *Self-efficacy* beliefs (F(1,310)=9.935, p=.003, Partial η^2=.028), as well as lower Prior *Extrinsically Determined Motivation* (F(1,310)=6.701, p=.010, Partial η^2=.021). Partial η^2 indicate that both of these effects were small. Pair-wise comparisons show that women were significantly lower (at the level .05) on self-efficacy (Mean Difference (women-men)=-.218) and on extrinsically determined motivation (Mean Difference (women-men)=-.199). Figures 5 and 6 above illustrate these differences.

Summary and Discussion

- There are no significant differences between the means of *High School Math Performance* across the three conditions.
- There are no significant differences between the means of Prior *Knowledge of Algebra* across the three conditions.
- The pre-instruction *Knowledge of Functions* is significantly lower in *Condition* C2, compared to both *Condition* C1 and *Condition* C3. There is no significant difference in pre-instruction knowledge of functions between *Condition* C1 and *Condition* C3.
- Prior *Knowledge of Algebra* and *Functions* and *High School Math Performance* do not differ by *Gender* or by the interaction *Condition***Gender*.
- There are no significant differences in the means of any of the pre-instruction motivation variables between *Condition* C1 and *Condition* C3.

Results

- Prior *Amotivation* is significantly higher in *Condition* C2 compared to *Condition* C1 as well as *Condition* C3.
- Prior *Self-determined Motivation* is significantly lower in *Condition* C2 in contrast with *Condition* C1.
- Women participating in the study have significantly lower Prior *Self-efficacy* beliefs, as well as lower *Self-determined Motivation* compared to their male peers.

The above results indicate that there were some differences between conditions, both in terms of prior knowledge, and in terms of student motivation. In all cases when the differences were found, the effect size of those differences was always small, explaining less than 10% of variance. There is one worrisome aspect: students in *Condition* C2 were consistently different from students in *Conditions* C1 and C3. Students in *Condition* C2 were not only less likely to possess knowledge of functions when they enrolled in the Calculus course, they were also more amotivated before the course began. Although the effect sizes were small, the cumulative effect of gaps in prior knowledge of functions and higher amotivation may need to be considered when we interpret the results of the analyses of differential impact of *Condition*s on student achievement and perseverance. There were also significant differences in self-determined motivation. The effect size was small (less then 3% of variance), and it was due to lower self-determined motivation of *Condition* C2 students in contrast with *Condition* C1 students. It is unlikely that differences in achievement and perseverance will be influenced by these differences in Prior *Self-determined Motivation*.

The gender differences that were found were predictable. Many researchers have found that women are less confident about their ability to do well in mathematics in comparison with men. Similarly, women in this sample were less self-efficacious than men. The effect size was small, but since self-efficacy often plays an important role in student academic performance, we will need to consider these differences when interpreting the results of the analyses of gender differences in achievement and perseverance. Also, men had higher extrinsically regulated motivation. Research on the impact of extrinsically regulated motivation on achievement is inconclusive because in many studies it has a negative impact while in others it has no impact or even positive impact.

Results

2. *Student Perceptions of Learning Environment*

In this study, there were nine classes which were taught by eight different instructors, each of which may have a different teaching style. The three experimental conditions imposed three different instructional strategies, but due to possibly different teaching styles we anticipated perhaps different learning environments in each section of the course.

The results of univariate GLM, with *Class* and *Gender* as independent variables and *Perceptions of Autonomy* as dependent variable, are shown in Table 4. below.

Table 4. Tests between subjects: *Perceptions of Autonomy* (N=230)

Source	df	F	Sig.	Partial η^2
Class	6	11.537	.000	.231
Gender	1	2.760	.098	.012
*Class * Gender*	6	.852	.531	.022

The table above indicates that students' perceptions of learning environments differed significantly from class to class ($F(6,230)=11.537$, $p<.001$, Partial $\eta^2=.231$), but the gender differences between the means of student perceptions of learning environment or the interaction *Class * Gender* were not significant. The graph in Figure 7 at the right illustrates differences between the means of *Perceptions of Autonomy*. Partial η^2 indicates that the effect size is large: the independent variable *Class* explains 23.1% of variance in student *Perceptions of Autonomy*. Perceptions of autonomy have been shown to have a positive impact on student achievement and perseverance, particularly for women (Dedic, *et al.*, 2007). Consequently, we anticipate that differences in students' achievement and perseverance could be also explained by differences in their perceptions of autonomy. Note that one teacher taught two sections and these two sections are reported as one class in Figure 7.

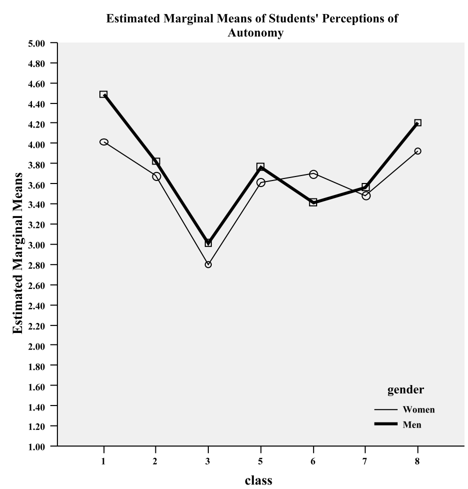

Figure 7

Results

Each experimental *Condition* consisted of three classes. To estimate the between condition differences in perceptions of the learning environment, we used univariate GLM with *Perception of Autonomy* as dependent variable and independent variables *Condition* (instead of *Class*) and *Gender*. Table 5 below shows the results of between subjects tests.

Table 5. Tests between subjects: *Perceptions of Autonomy* (N=230)

Source	df	F	Sig.	Partial η^2
Condition	2	2.532	.082	.021
Gender	1	1.381	.241	.006
*Condition * Gender*	2	.276	.759	.002

The above results indicate that *Condition*, *Gender* or interaction of *Condition * Gender* did not significantly affect students' *Perception of Learning Environment*.

<u>Summary and Discussion</u>

- Students' *Perception of Learning Environment* differs significantly from class to class.
- *Gender* and the *Gender * Class* interaction had no impact on *Perception of Learning Environment*.
- When the classes were agglomerated into conditions, no significant difference in the means of *Perception of Learning Environment* was found between the three conditions.

It appears that students' perception of autonomy is more an effect related to the particular teacher rather than an effect related to the experimental conditions. Therefore, we will refer to the above results in our discussion of outcomes, but we will not include *Perception of Autonomy* as a covariate in subsequent analysis.

3. *Changes in Algebra and Functions Knowledge*

Although not a primary focus of a Calculus class, both algebra and functions knowledge play a large role in the learning of Calculus ideas and techniques. Therefore, it is interesting to measure any changes in algebra and functions knowledge that take place during Calculus courses. To appraise these changes, while taking into account any differences in prior knowledge, we estimated a multivariate GLM with the following two post-instruction scores, Post *Knowledge of Algebra* and Post *Knowledge of Functions*, as dependent variables, *Condition* and *Gender* as independent variables, and Prior *Knowledge of Algebra* and Prior *Knowledge of Functions* as covariates. The results are presented in Table 6 below.

Results

Table 6. Multivariate tests: Post knowledge of algebra and functions (N=218)

	Hypothesis df	F	Sig.	Partial η^2
Intercept	2	54.96	.000	.345
Prior *Knowledge of Algebra*	2	48.81	.000	.318
Prior *Knowledge of Functions*	2	13.28	.000	.113
Condition	4	4.269	.000	.039
Gender	2	1.084	.340	.010
Condition * *Gender*	4	.162	.957	.002

Not surprisingly, pre-instruction algebra and functions knowledge significantly influence post-instruction algebra and functions knowledge. Post-instruction knowledge is also found to be significantly dissimilar across the three experimental conditions.

The estimated marginal means for post-instruction algebra and functions knowledge are reported in Table 7 below.

Table 7. Estimated marginal means of post knowledge of algebra and functions

				95% Confidence Interval	
Dependent Variable	*Condition*	Mean	Std. Error	Lower Bound	Upper Bound
Post *Knowledge of Algebra*	C1	.407	.037	.334	.480
	C2	.365	.035	.295	.434
	C3	.386	.035	.317	.455
Post *Knowledge of Functions*	C1	.502	.046	.410	.593
	C2	.315	.044	.228	.402
	C3	.563	.044	.477	.649

Figures 8 & 9 below illustrate the differences in post knowledge of algebra and functions.

Results

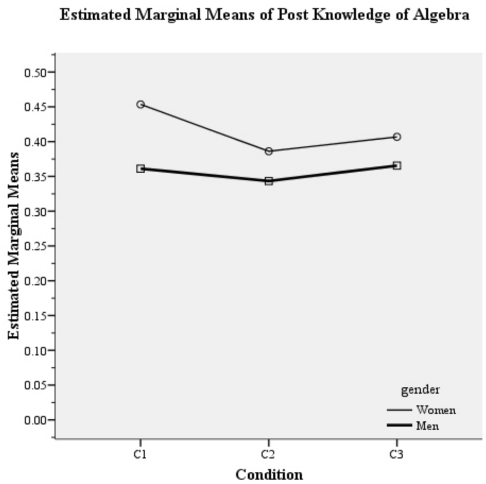

Figure 8

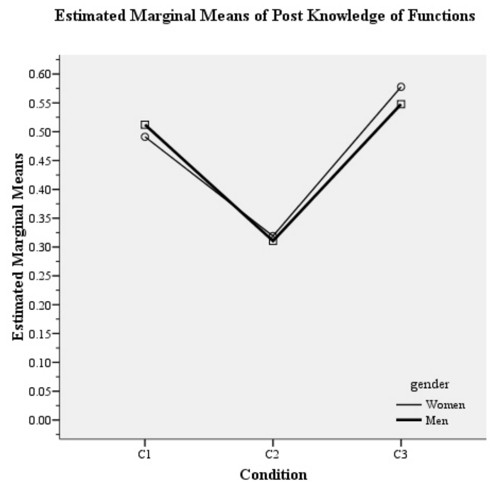

Figure 9

Pair-wise comparisons indicate that students in *Condition* C2 had significantly lower (at the level .05) score on *Knowledge of Functions* than students in *Condition* C1 (Mean Difference (C2-C1)=-.187) and students in *Condition* C3 (Mean Difference (C2-C3)=-.248). There were no significant gender differences in post-instruction knowledge of algebra and functions. Similarly, the *Condition* * *Gender* interaction had no significant impact on post-instruction knowledge of algebra and functions.

Further evidence for the external validity of the algebra and functions tests is furnished by the correlations between the pre and post-test scores and the variables capturing knowledge and achievement at both the high school level and in Calculus. These correlations are reported in Table 8 below:

Table 8. Correlations between knowledge of algebra and functions and achievement in Calculus

Variable	*fscore*	HSMP	PRKA	PSKA	PRKF
High School Math Performance	.483				
PRior *Knowledge of Algebra*	.323	.409			
PoSt *Knowledge of Algebra*	.416	.435	.564		
PRior *Knowledge of Functions*	.248	.341	.193	.269	
PoSt *Knowledge of Functions*	.545	.237	.277	.420	.429

All correlations in the above table are significant at $p<.01$.

Results

The high correlations between the scores on the algebra and functions tests and student achievement in high school (*High School Math Performance*) and in the Calculus course (*fscore*), indicate the external validity of these instruments. Note the high correlation between students' knowledge of functions on post-test and *fscore*. This is not surprising since functions are the main objects of study in Differential Calculus. Although students practice algebra while computing in Calculus, the subject of algebra is not directly taught. Hence, we see the somewhat lower, but still high, correlation coefficient between Post *Knowledge of Algebra* and achievement in Calculus (*fscore*).

<u>Summary and Discussion</u>

- There are no significant differences in the post-instruction algebra skills in the three conditions.
- Students in *Condition* C1 and *Condition* C3 have a significantly higher level of understanding of functions compared to students in *Condition* C2, even after correcting for differences in prior knowledge.
- There are no significant differences in post-instruction knowledge of algebra and functions by *Gender*, or due to an interaction of *Gender* and *Condition*.
- There are high correlations between algebra skills and functions knowledge on the one hand and achievement on the other hand, both in high-school and in Calculus.

The slight increases in algebra knowledge in all classes are not unexpected, given that algebra skills practice is present in any Calculus class. These increases were uniform across the conditions. However, increases in functions knowledge were more substantial. This could be attributed to the fact that all instructors spent time (the same amount in each class) reviewing knowledge of functions. The students in C2 who had inferior prior knowledge of functions also improved the least, even in the model where there was a correction for prior-knowledge. One possible explanation is that although the correction in the model is linear, prior knowledge has a stronger than linear effect on acquisition of further knowledge. Another possible explanation is that the learning environment in *Condition* C2 somehow provided the students with the least opportunity to improve their knowledge of functions.

4. *Student Motivation*

In the analysis in this section we address two questions. Did the experimental condition affect students' motivation and students' self-efficacy beliefs? Was there an interaction between

Results

Gender and *Condition* in terms of changes in students' motivation? We hypothesized that having feedback during learning, students' motivation and self-efficacy would rise, particularly amongst men.

To assess pre/post changes in motivation and self-efficacy, a multivariate analysis of variance was used. The dependent variables were Post *Self-efficacy*, Post *Self-determined Motivation*, Post *Extrinsically Determined Motivation* and Post *Amotivation*, while the independent variables were *Condition* and *Gender*, and the covariates were Prior *Self-efficacy*, Prior *Self-determined Motivation*, Prior *Extrinsically Determined Motivation* and Prior *Amotivation*.

Table 9. Multivariate Test: Post *Self-efficacy* and Motivation (N=225)

Effect	F	Hypothesis df	Sig.	Partial η^2
Intercept	19.293	4	.000	.267
Prior *Self-efficacy*	13.319	4	.000	.201
Prior *Self-determined Motivation*	23.100	4	.000	.304
Prior *Extrinsically Determined Motivation*	42.611	4	.000	.446
Prior *Amotivation*	6.912	4	.000	.115
Condition	1.084	8	.373	.020
Gender	1.354	4	.251	.025
*Condition * Gender*	1.621	8	.117	.030

The multivariate tests showed no significant main or interaction effects on combined DVs. The tests of between subjects show a significant effect of *Condition* on Post Self-efficacy (F(2,215)=3.439, p=.034, Partial η^2=.031). Pair-wise comparison indicates that students in *Condition* C2 had significantly lower (at the level .05) self-efficacy beliefs than students in *Condition* C3 (Mean Difference (C2-C3)=-.281). In addition, there was a significant effect of interaction between *Condition* and *Gender* on Post *Self-efficacy* (F(2,215)=3.052, p=.049, Partial η^2=.028). In both cases the associations were very weak, as the values of Partial η^2 indicate. No other significant effects were observed. The graph in the Figure 10 shows the plot of the estimated means of Post *Self-efficacy*.

Results

The graph in Figure 10 shows that women's Post *Self-efficacy* as being lower than that of men in *Conditions* C1 and C2. Although the differences between the means are not significant, this result is in agreement with results reported in many studies (*e.g.,* Dedic *et al.*, 2004) of gender differences in self-efficacy in mathematics courses. We observe that men's self-efficacy is the same across all three conditions. Thus, the fact that on average students in *Condition* C3 believed in their competence significantly more that students in *Condition* C2 did, is caused by the heightened self-efficacy of women in *Condition* C3. Since interactive sessions were the only characteristic that distinguished the learning environment in *Condition* C3 from that in *Condition* C2, we would speculate that these interactive sessions promoted the self-efficacy of women.

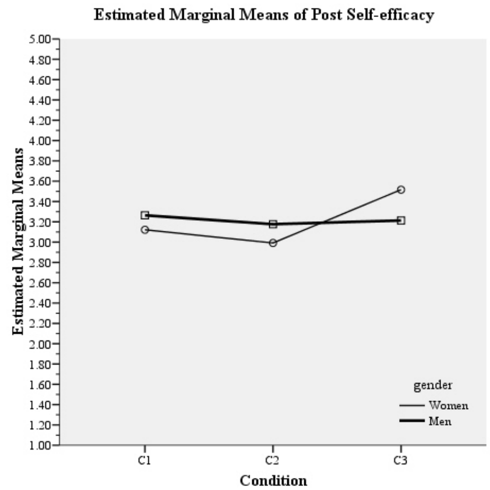

Figure 10

We also examined how students' motivation related to two outcome variables: student achievement (*fscore*) and the probability of taking further courses in mathematics (*Probability*). To this end we determined the correlations between motivational variables and the two outcome variables. Table 10 below shows the results of these calculations. The pre-instruction motivational variables do not correlate strongly with either achievement or the probability of taking further mathematics courses. On the other hand, with the exception of Post *Extrinsically Determined Motivation*, all other post-instruction motivational variables significantly correlate with outcome variables. These correlations may need to be taken into account in further analyses. Note that ** indicate that the correlation coefficient is significant at p=.01 and * indicates the p-value .05.

Results

Table 10. Correlation between self-efficacy, motivation and student achievement in Calculus.

	fscore	Prob	PS	PSM	PEDM	PA	PRS	PRSM	PREDM
Prob.	.937**								
Post *Self-efficacy*	.596**	.617**							
Post *Self-determined Motivation*	.399**	.363**	.538**						
Post *Extrinsically Determined Motivation*	-.037	-.061	.053	.362**					
Post *Amotivation*	-.462**	-.432**	-.573**	-.672**	-.140*				
Prior *Self-efficacy*	.183**	.176**	.528**	.320**	.018	-.369**			
Prior *Self-determined Motivation*	.136*	.109	.204**	.561**	.105	-.394**	.383**		
Prior *Extrinsically Determined Motivation*	-.099	-.113	-.053	.159*	.639**	.074	.058	.243**	
Prior *Amotivation*	-.158**	-.158**	-.363**	-.344**	.017	.503**	-.524**	-.575**	.002

Summary and Discussion

- There are no significant differences on post-instruction motivational variables by *Condition* or by *Gender*.
- Women's post-instruction self-efficacy was lower than that of men in *Condition* C1 and *Condition* C2, but higher than that of men in *Condition* C3.
- Pre-instruction motivational variables did not strongly correlate with achievement and persistence variables, while post-instruction motivational variables did correlate strongly with these two outcome variables.

5. *Students' Achievement and Perseverance*

In this analysis we address two main research questions: 1. Do the experimental conditions impact on students' achievement and perseverance? and 2. Are there gender differences in students' achievement and perseverance, and if so, are those differences affected by the experimental conditions?

We hypothesized that since WeBWorK assignments provide feedback while students are learning, students' achievement and perseverance would be higher in *Condition* C2 than in *Condition* C1. We also hypothesized that because of additional instructional support in experimental *Condition* C3, students in *Condition* C3 would outperform students in *Conditions* C1 and C2, and that students in *Condition* C3 would persevere more. In addition, we hypothesized that instructional support would impact men and women differently.

Results

This analysis was carried out in three steps:

1. the dependent variables were student achievement (*fscore*) and student perseverance in terms of the probability of taking another course in mathematics (*Probability*) and the independent variables were *Condition* and *Gender,* **and no covariates were included**;
2. the dependent variables were *fscore* and *Probability,* the independent variables were *Condition* and *Gender,* **and covariate were included**;
3. a step-down analysis was carried out, **and covariate were included**.

<u>Step1</u>.

Table 11. Multivariate Tests: Students' achievement and perseverance (N=317)

Effect	Hypothesis df	F	Sig.	Partial η^2
Intercept	2	1490.293	.000	.906
Condition	4	16.436	.000	.096
Gender	2	4.066	.018	.026
*Condition * Gender*	4	.475	.754	.003

The results in Table 11 show that *Condition* significantly affects a linear combination of dependent variables (student achievement and probability of taking further courses in mathematics), although the association is moderate (Partial η^2=.096). *Gender* also significantly affects combined DVs, but the association is very small (Partial η^2=.026). The interaction between independent variables is not significant. The graph in Figure 11 at the right shows a plot of estimated marginal means of *fscore* (student achievement).

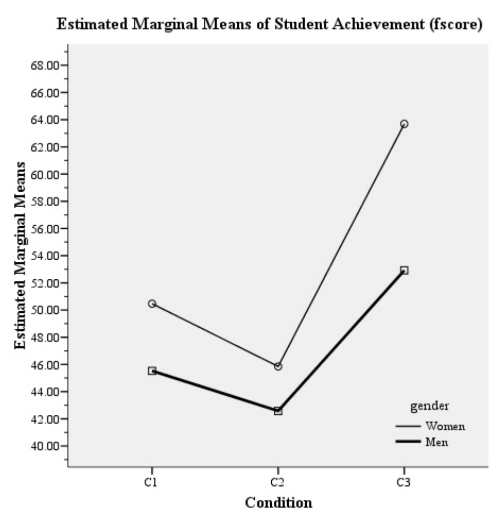

Figure 11

Using Bonferroni adjustment for multiple comparisons, we found that students in *Condition* C3 significantly outperformed students in *Condition* C1 (Mean Difference (C3-C1)=10.308), as well as students in *Condition* C2 (Mean Difference (C3-C2)=14.089). These differences are significant at the .05 level. The difference between achievement of students in *Condition*s C1

Results

and C2 is not significant. In addition, women significantly outperformed men (Mean Difference (Women-Men)=6.328). Again, this difference is significant at the .05 level.

The graph shown in Figure 12 plots estimated marginal mean of probability of taking more advanced mathematics courses. Pair-wise comparisons revealed that students in *Condition* C3 were more likely to take more advanced courses in mathematics than students in *Condition* C1 (Mean Difference (C3-C1)=.124). Also, women in all conditions were more likely than men to take more advanced courses in mathematics (Mean Difference (Women-Men)=.124). These differences are significant at the .05 level.

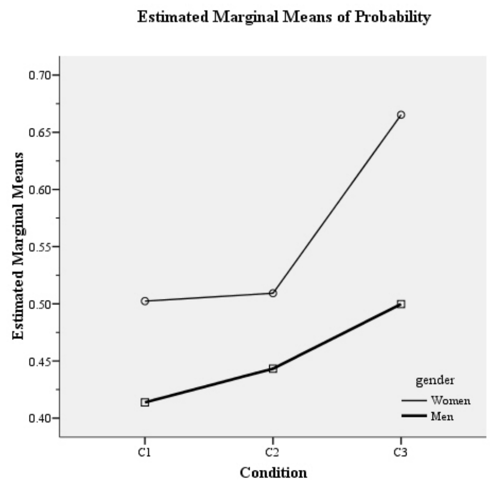

Figure 12

<u>Step 2</u>.

We anticipated that both dependent variables would also be affected by students' prior achievement and motivation. Therefore, variables (*High School Math Performance*, Prior *Knowledge of Algebra* and Prior *Knowledge of Functions*) were included as covariates as we further investigated how *Condition* and *Gender* affect students' achievement and perseverance. In addition, because we have shown that students' achievement and perseverance is correlated with their motivation, we also included Post *Self-efficacy*, Post *Self-determined Motivation*, Post *Extrinsically Determined Motivation* and Post *Amotivation* as covariates. Table 12 below shows the results of multivariate tests.

Results

Table 12. Multivariate Tests: Achievement and Perseverance. (N=305)

Effect	Hypothesis df	F	Sig.	Partial η^2
Intercept	2	9.721	.000	.092
Post *Self-efficacy*	2	24.497	.000	.204
Post *Self-determined Motivation*	2	.044	.957	.000
Post *Extrinsically Determined Motivation*	2	1.602	.204	.016
Post *Amotivation*	2	2.164	.118	.022
High School Math Performance	2	15.003	.000	.136
Prior *Knowledge of Algebra*	2	.102	.903	.001
Prior *Knowledge of Functions*	2	.011	.989	.000
Condition	4	14.915	.000	.135
Gender	2	3.755	.025	.038
*Condition * Gender*	4	.256	.906	.003

The above table shows that both main effects are significant, while the interaction between *Condition* and *Gender* is not. We found that *Condition* affected the linear combination of *fscore* (students' achievement) and *Probability* ($F(2,192)=14.915$, $p<.001$, Partial $\eta^2=.135$). This result reflected a modest association between *Condition* and the combined DVs. *Gender* also affected the combined DVs ($F(1,192)=3.755$, $p=.025$, Partial $\eta^2=.038$), but the association was small.

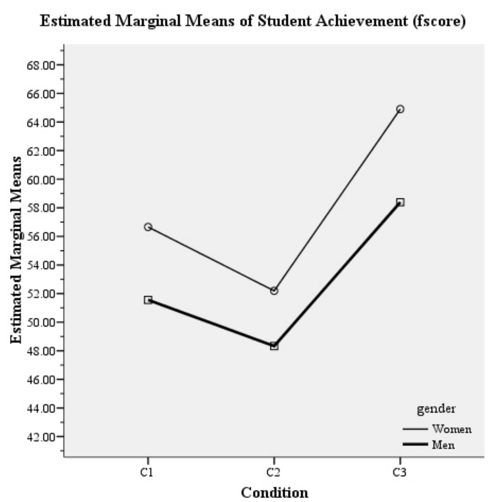

Figure 13

The graph in Figure 13 above shows how students' achievement varied in different conditions and also how women outperformed men in all conditions. We have run pair-wise comparison, using Bonferroni adjustment, to see how students' achievement varied in each of the conditions. Table 13 below shows the results of this comparison.

Results

Table 13. Pair-wise Comparisons: Students' Achievement (N=305)

Dependent Variable	(I) *Condition*	(J) *Condition*	Mean Difference (I-J)	Sig.	95% Confidence Interval for Difference	
					Lower Bound	Upper Bound
fscore	C1	C2	3.848	.375	-2.186	9.883
		C3	-7.540*	.007	-13.44	-1.641
	C2	C3	-11.389*	.000	-16.651	-6.126

The table above shows that students in *Condition* C3 significantly outperformed students in *Condition* C1 (Mean Difference (C3-C1)=7.540) and students in *Condition* C2 (Mean Difference (C3-C2)=11.389), with more than 95% confidence that neither of these differences is equal to zero. Similarly, pair-wise comparison was run for gender differences in students' achievement. Women outperformed men (Mean Difference (Women-Men)=5.159) with more than 95% confidence that this difference between *Gender* means is not equal to zero.

The graph in Figure 14 on the right shows differences in students' perseverance in terms of the probability of taking more advanced courses in mathematics (*Probability*). The graph indicates that while women were more likely to take advanced courses in mathematics than men, the differences between the conditions were not obvious. Using Bonferroni adjustment, the pair-wise comparison revealed that indeed there were no significant differences between the conditions. It also showed, with 95% confidence, that in terms of probability of taking more advanced courses (Mean Difference (Women-Men)=.084), the probability of a woman taking further courses in mathematics is 8.4% higher than the probability of a man doing so.

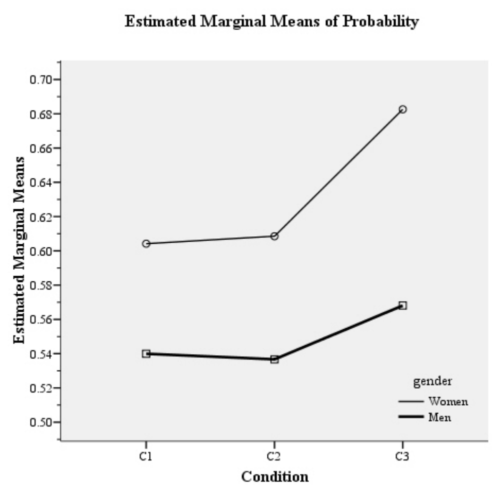

Figure 14

Results

Step 3.

From the data in Table 12. we also observe that only Post *Self-efficacy* and *High School Math Performance* as covariates affect combined DVs. The other covariate variables (Post *Self-determined Motivation*, Post *Extrinsically Determined Motivation* and Post *Amotivation*) do not significantly affect the linear combination of *fscore* and *Probability*. To confirm this observation, and to further investigate the impact of covariates on each of the dependent variables, we ran a multiple regression for *fscore* with all covariates as predictors. The β value of .314 for *High School Math Performance* was significantly different from zero ($t(218)=6.007$, $p<.001$) and similarly, the β value of .399 for Post *Self-efficacy* was also significantly different from zero ($t(218)=6.344$, $p<.001$). We then ran a multiple regression for *Probability* with *fscore* and all the other covariates as predictors. The β value of .823 for *fscore* was significantly different from zero ($t(218)=24.475$, $p<.001$) and also the β value of .094 for *High School Math Performance* was significantly different from zero ($t(218)=3.358$, $p=.001$). Similarly, the β value of .111 for Post *Self-efficacy* was significantly different from zero ($t(218)=3.256$, $p=.001$). All the other βs were not significantly different from zero.

The above results warranted further investigation of the relationship between dependent variables and independent variables. To this end, we conducted a univariate analysis of variance with *fscore* as dependent variable, *Condition* and *Gender* as independent variables with covariates (*High School Math Performance*, Prior *Knowledge of Algebra*, Prior *Knowledge of Functions*, Post *Self-efficacy*, Post *Self-determined Motivation*, Post *Extrinsically Determined Motivation* and Post *Amotivation*). Table 14 below shows the results of this analysis.

Table 14. Impact of *Condition* and *Gender* on Student Achievement (*fscore*). (N=305)

Source	df	F	Sig.	Partial η^2
Post *Self-efficacy*	1	38.536	.000	.167
Post *Self-determined Motivation*	1	.029	.864	.000
Post *Extrinsically Determined Motivation*	1	.502	.479	.003
Post *Amotivation*	1	3.239	.073	.017
High School Math Performance	1	24.370	.000	.113
Prior *Knowledge of Algebra*	1	.048	.827	.000
Prior *Knowledge of Functions*	1	.002	.966	.000
Condition	2	14.142	.000	.128
Gender	1	7.085	.008	.036
*Conditions * Gender*	2	.188	.829	.002

Results

The above table indicates that *Condition* (p<.001, Partial η^2=.128) significantly affects students' achievement, and that the association between *Condition* and *fscore* is moderate. *Gender* also significantly affects *fscore* (p=.008, Partial η^2=.036), but the association between *Gender* and *fscore* is small. The interaction between *Condition* and *Gender* does not influence *fscore*. Notice that students' achievement was also significantly associated with *High School Math Performance* (p<.001, Partial η^2=.113) and Post *Self-efficacy* (p<.001, Partial η^2=.167).

Next we ran a univariate analysis of variance with *Probability* as dependent variable, *Condition* and *Gender* as independent variables, and *fscore* included among the covariates. In this manner we studied the impact of *Condition* and *Gender* on *Probability*, while controlling for differences in *fscore*. The results of this analysis are shown in Table 15 below.

Table 15. Students' perseverance (N=305)

Source	df	F	Sig.	Partial η^2
Post *Self-efficacy*	1	8.877	.003	.044
Post *Self-determined Motivation*	1	.059	.808	.000
Post *Extrinsically Determined Motivation*	1	2.697	.102	.014
Post *Amotivation*	1	1.087	.298	.006
High School Math Performance	1	5.114	.025	.026
Prior *Knowledge of Algebra*	1	.156	.693	.001
Prior *Knowledge of Functions*	1	.021	.884	.000
fscore	1	561.571	.000	.746
Condition	2	15.768	.000	.142
Gender	1	.445	.505	.002
*Condition * Gender*	2	.325	.723	.003

Results

These results show that when we control for *fscore*, only *Condition* significantly affects students' perseverance. A plot of marginal means of *Probability* is shown in Figure 15. Pair-wise comparisons show that, when adjusted for *fscore*, students in *Condition* C2 were more likely to take further math courses than students both in *Condition* C1 (Mean Difference of *Probability* (C2-C1)=.055) and in *Condition* C3 (Mean Difference of *Probability* (C2-C3)=.108). These mean differences are significant at the .05 level.

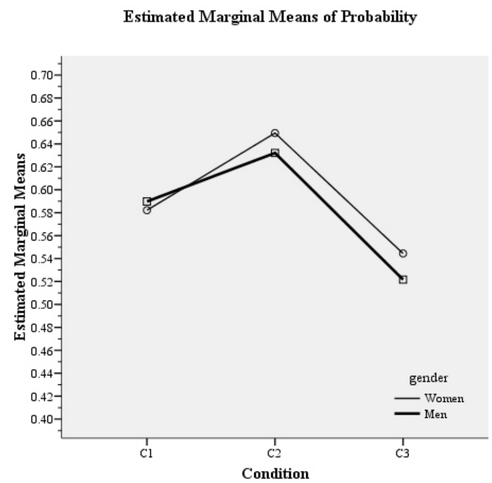

Figure 15

<u>Summary and Discussion</u>

- Without a correction for the differences in prior knowledge or motivation, on the achievement variable *fscore*, students in *Condition* C3 significantly outperformed students in both *Condition* C1 and *Condition* C2. No significant difference was found on *fscore* between students in *Condition* C1 and those in *Condition* C2. Women significantly outperformed men on *fscore.*
- Without correction for prior differences we determined that students in *Condition* C3 are more likely to take further math classes than students in *Condition* C1, and that women are more likely than men to take further math classes.
- Considering the achievement variable *fscore*, with correction for differences in prior achievement and knowledge, as well as correction for differences in motivation, students from *Condition* C3 outperformed students in *Condition* C1 by a mean difference of 7.54 points. Further, students from *Condition* C3 outperformed students in *Condition* C2 by a mean difference of 11.39 points and women outperformed men by a mean difference of 5.16 points. All of these differences are significant at .05 level.
- Considering the probability of taking further math courses, with correction for individual differences in prior achievement and knowledge, as well as motivation, *Condition* did not significantly affect the likelihood of taking more math courses.; women were, on average, 8.4% more likely to take further math courses than men.
- Univariate tests on the achievement variable *fscore* show that it is significantly affected by:

Results

Condition (Partial η^2=.128); *Gender* (Partial η^2=.036); *High-School Math Performance* (Partial η^2=.113) and Post *Self-efficacy* (Partial η^2=.167). Partial η^2 shows moderate associations between variables in each case.

- Univariate tests on the perseverance variable *Probability*, with *fscore* amongst the covariates, show that *Condition* significantly affects *Probability*.

The answer to the two main experimental questions appears to be affirmative. With no correction for individual differences, *Condition* significantly affects students' achievement (*fscore*) and perseverance (*Probability*) and there is a significant impact on both achievement and perseverance due to *Gender*.

In subsequent analysis, which accounted for individual differences in prior achievement and knowledge, as well as self-efficacy and motivation, it was found that students in *Condition* C3 outperformed students in *Condition*s C1 and C2, and women outperformed men in terms of achievement. There was a significant impact of *Condition* on perseverance, and women are more likely to persevere. We have shown, as hypothesized, that students in *Condition* C3 outperform students in *Condition* C1 and *Condition* C2 in achievement and perseverance, but the results failed to show that students in *Condition* C2 outperform students in *Condition* C1.

Next we conducted a step-down analysis by studying the impact of independent variables on *fscore* alone, and determined that students in *Condition* C3 outperformed students in *Conditions* C1 and C2 and women outperformed men. In addition, we still did not observe any differences between students in *Conditions* C1 and C2, and the interaction between *Condition* and *Gender* did not have a significant impact on achievement. In the next step of the step down analysis, we examined the impact of *Condition*, *Gender* and *Condition* * *Gender* on students' perseverance, while controlling for *fscore*. It was found that there were no significant effects of *Gender* or *Condition* * *Gender* on students' perseverance, with only *Condition* significantly affecting perseverance. Students in *Condition* C2 "seemed" to be more likely to persevere than students in *Conditions* C1 and C3, however this result will be discussed in more detail below.

6. *Impact of Experimental Conditions on Students' Grades and Effort*

In this analysis we examined the impact of the three experimental conditions on students' Achievement (*fgrade*) and students' Effort (*Assignment, Frequency*).

Results

Table 16 below shows the results of a multivariate analysis with teacher assigned final grade (*fgrade*) and *Frequency* as dependent variables, *Condition* and *Gender* as independent variables, and with Students' prior knowledge and achievement and motivational characteristics serving as covariates.

Table 16. Multivariate Tests: Students Grades and Frequency (N=307)

Effect	Hypothesis df	F	Sig.	Partial η^2
Intercept	2	27.681	.000	.223
Post *Self-efficacy*	2	26.425	.000	.215
Post *Self-determined Motivation*	2	.051	.951	.001
Post *Extrinsically Determined Motivation*	2	.692	.502	.007
Post *Amotivation*	2	.665	.516	.007
High School Math Performance	2	18.571	.000	.161
Prior *Knowledge of Algebra*	2	.008	.992	.000
Prior *Knowledge of Functions*	2	1.077	.343	.011
Condition	4	3.263	.012	.033
Gender	2	4.507	.012	.045
*Condition * Gender*	4	.365	.834	.004

We determined that *Condition* significantly affects a linear combinations of dependent variables (F(2,193)=3.263, p=.012, Partial η^2=.033), but the association between *Condition* and the combined dependent variables is weak. Similarly, *Gender* significantly affects combined dependent variables (F(1,193)=4.507, p=.012, Partial η^2=.045), but again the association is weak. In addition, there is no significant impact on combined dependent variables due to an interaction between *Condition* and *Gender*. The graphs in Figure 16 illustrate gender differences in *fgrade*.

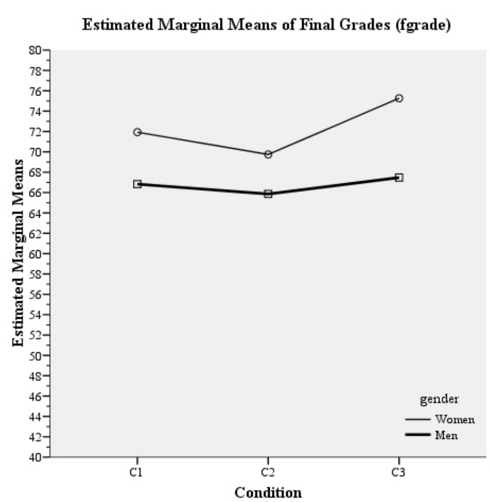

Figure 16

To further investigate how *Condition* and *Gender* affect final grades, we ran pair-wise comparisons, which showed that there were no significant differences between the means of final

Results

grades in the three conditions. However, women outperformed men (Mean Difference (Women - Men)=5.594), and this difference is significant at the .05 level.

To examine how *Gender* affects students' effort, we ran pair-wise comparisons on *Frequency*, which showed that students in *Condition* C3 submitted assignment significantly more frequently than students in *Condition* C2 (Mean Difference (C3 - C2)=.764), and this difference is significant at the .05 level. The graph in Figure 17 illustrates these differences. However, it should be noted that there were 10 assignments altogether, and that students in all *Condition*s submitted on average close to 9 out of 10 of them.

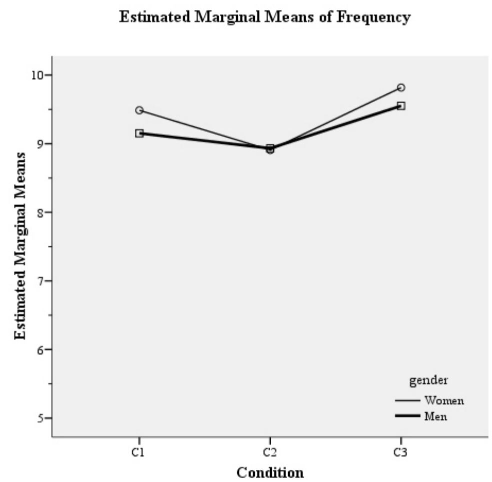

Figure 17

Analysis of students' performance on assignments (*Assignment*) revealed that *Condition* significantly affected students' performance on assignments ($F(2,348)=20.941$, $p<.001$, Partial $\eta^2=.107$), showing a modest association between *Condition* and students' grades on assignments (*Assignment*). *Gender* also significantly affected students' grades on assignments ($F(2,348)=7.048$, $p=.008$, Partial $\eta^2=.020$), showing a small association between *Gender* and the grade on assignments. There was no significant impact of interaction between *Condition* and *Gender* on assignment grades. Pair-wise comparisons revealed that students in *Condition* C3 outperformed students in *Condition* C1 on assignments (Mean Difference (C3 - C1)=15.274) as well as students in *Condition* C2 (Mean Difference (C3 - C2)=16.843), while there was no significant difference between the assignment

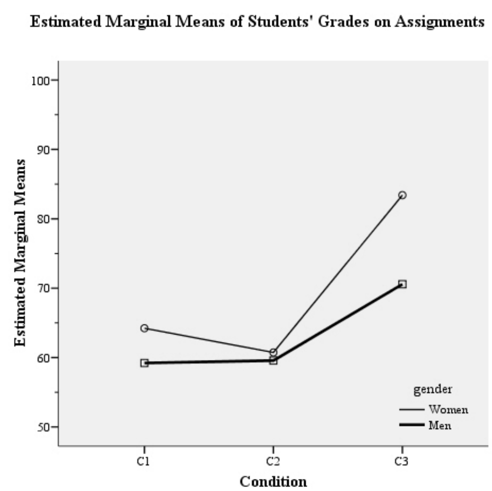

Figure 18

grades of students in *Condition*s C1 and C2. Also, women outperformed men on assignments (Mean Difference (Women - Men)=6.338). All differences are significant at the .05 level. The graph in Figure 18 above shows these differences in assignment grades.

The results above indicate that there were no significant differences in the means of final grades across the three conditions. This result contradicts the results shown in the previous section where we found significant differences between the means of student achievement as measured by *fscore* in the three conditions. Naturally we determined to investigate this discrepancy. We assumed that final grades (*fgrade*), computed by instructors, should be related to the assessment of knowledge of Calculus (*fscore*) by independent coders. Therefore, we computed linear regression coefficients for each of the conditions, with *fgrade* as the dependent variable, and *fscore* as the independent variable. The resulting regression equations are shown in Table 17 below.

Table 17. Regression of final grades (*fgrade*) vs. student achievement (*fscore*).

Condition	fgrade=B(SD)*fscore + CONSTANT(SD)	β	t	Sig.
C1	fgrade=1.068(.033)*fscore + 7.758(1.744)	.948	31.906	<.001
C2	fgrade=1.054(.031)*fscore + 13.516(1.528)	.956	34.283	<.001
C3	fgrade=1.006(.030)*fscore + 8.477(1.841)	.953	33.492	<.001

The t-test statistics indicate that the probability of standardized βs being equal to zero is less than .001 in all conditions. Note that the slope parameter is nearly 1 in all conditions, indicating that instructors and independent coders were remarkably consistent in assessing students' performance. However, the constant coefficient varied across the three conditions. *Condition* C2, with the lowest mean on *fscore*, also had the highest constant value. It appears that *Condition* C2 instructors increased grades more than instructors in either *Condition* C1 or *Condition* C3.

To test the significance of these differences, we ran ANCOVA, with *fgrade* as the dependent variable, *Condition* and *Gender* as independent variables, and *fscore* as covariate. The results show that *Condition* significantly affected final grades ($F(2,339)=31.927$, $p<.001$, Partial $\eta^2=.159$), while *Gender* and interaction between *Condition* and *Gender* did not have a significant effect. This means that there is a modest association between *Condition* and *fgrade*. Pair-wise comparison shows that when corrected for *fscore*, there was a significant difference in mean final

Results

grades between students in *Condition* C3 and *Condition* C1 (Mean Difference (C3-C1)=-2.891), as well as between students in *Condition* C3 and *Condition* C2 (Mean Difference (C3-C2)=-8.082), and even between students in *Condition* C1 and *Condition* C2 (Mean Difference (C1-C2)=-5.191). All differences are significant at .05 level. This confirms that *Condition* C2 instructors increased the grades the most and *Condition* C3 instructors raised the grades the least, with *Condition* C1 instructors being in-between these two extremes. It should be noted that in pair-wise comparison, the phrase "Mean Differences" refer to differences between estimated marginal means which were computed by adjusting for individual differences in *fscore*. In other words, the marginal mean for any one of the conditions is computed as though all students had the same *fscore*, equal to the average *fscore* of the whole sample. Table 18 below shows estimated marginal *fgrade* to illustrate the differences we have listed above.

Table 18. Marginal mean estimates of grade, assuming average *fscore* = 49.557.

Condition	C1	C2	C3
est. *fgrade*	60.947	66.138	58.036

Failure rates also differed significantly across the three *Conditions* (Pearson $\chi^2 = .022$). In *Condition* C1, 43.2% failed, while only 36.0% of *Condition* C2 students failed, and 26.2% of *Condition* C3 students failed. In addition, there were significant *Gender* differences in failure rates (Pearson $\chi^2 = .038$), with 28.9% of women failing and 39.5% of men failing the Calculus course.

Post-experiment interviews with instructors generated some interesting observations. With the exception of one instructor, who retired the next year, and another who plans to retire shortly, all instructors now use WeBWorK in all of their courses. In addition, as result of this experiment, all instructors said that they plan to use the *Condition* C3 instructional strategy. Instructors teaching in *Condition* C1 expressed concerns that many of their students did not really work on assignments, but instead copied solutions from more diligent peers, something that would not be possible in *Condition* C2 and *Condition* C3, where problems contain components randomized by WeBWorK. Instructors teaching in *Condition* C1 and *Condition* C2 reported that students rarely sought help outside of class. On the other hand, instructors teaching in *Condition* C3 reported a deluge of e-mails sent by students asking questions about assignments. One such instructor discouraged e-mails, but invited students to discuss their questions with him face-to-face, either during computer lab classes or in his office. It appears that students in *Condition* C3 were seeking help outside of the classroom more actively than students in *Condition*s C1 and C2, and this despite the fact that students in *Condition* C3 already had extra instructor and peer support during their weekly in-class interactive sessions. One might be tempted to characterize this behaviour with the old maxim "give them an inch and they will take a yard", but of course, most instructors actively seek ways of getting students to come to ask questions, so the instructors in *Condition* C3 were pleased.

Summary and Discussion

- There are no significant differences between the final grades in the three conditions, but women significantly outperformed men.
- Final grades, *fgrade*, were boosted by all instructors as compared to independent coder assessment of student achievement, *fscore*. Instructors in *Condition* C2 boosted grades the most.
- Students in *Condition* C3 submitted assignments significantly more frequently than students in *Condition* C2.
- Students in *Condition* C3 significantly outperformed both those in *Condition* C1 and *Condition* C2 on assignment grades, while there was no significant difference between those in *Condition* C1 and *Condition* C2. Women significantly outperformed men on assignment grades.

We have shown that there were no significant differences in mean final grades across the three conditions. However, a closer examination of the results revealed that this apparent lack of impact was caused by instructors raising grades. The final grades of the poorest performing group of students, *Condition* C2, were increased the most. In absolute terms (see Table 18 above), their grades were increased by 16.581 percentage points. Students in *Condition* C1 saw their final grades increased by 11.390 percentage points, while students in *Condition* C3 had their final grades raised 8.479 percentage points. This result renders dubious the significant differences in failure rates across the conditions. Although students in *Condition* C2 had a lower failure rate when contrasted with students in *Condition* C1, this effect is probably largely due to the fact that instructors in *Condition* C2 raised grades more than those teaching in *Condition* C1, and this was confirmed by the results of ANCOVA. Thus, the lower failure rates in *Condition* C2 in comparison with *Condition* C1 seem to be an artifact of the instructors' grading policies. On the other hand, ANCOVA indicates that instructors teaching in *Condition* C3 increased final

Results

grades significantly less than instructors in *Condition* C1 (Mean Difference in final grades (C1-C3)=2.891) and also significantly less than instructors in *Condition* C2 (Mean Difference in final grades (C2-C3)=8.802). In this context the lower failure rates in *Condition* C3 when contrasted with failure rates in *Conditions* C1 and C2 become even more noteworthy.

The results show that students in all conditions submitted close to 90% of all assignments. Our comparison between the conditions corroborates the instructors' observations that students in *Condition* C3 exerted more effort. They submitted assignments significantly more frequently than students in *Condition* C2. We may also speculate that observations of instructors teaching in *Condition* C1 that students simply copied assignments may explain why there are no significant differences between students in *Condition* C3 and students in *Condition* C1 in terms of the frequency of submitted assignments. The instructional support in *Condition* C3 likely explains why *Condition* C3 students outperformed students in the two other *Conditions* on assignments, as well as in terms of achievement and perseverance.

Discussion

VII. Discussion

Since there were no significant differences in prior academic performance in mathematics between students in the three conditions, and since prior performance is usually the most reliable predictor of future performance, it is reasonable to attribute post-result differences to the differences amongst the three conditions. Additionally, in this quasi-experimental study we attempted to avoid pitfalls found in many studies of the effectiveness of Computer Aided Instruction (CAI), namely failure to control instructional design differences between control and experimental conditions (Jenks and Springer, 2002). Since in this study all instructors used the same text, assignments sets and evaluation schema of students' performance, we conclude that the mode of delivery of assignments (paper vs. WeBWorK), and consequent promptness of feedback (one week later vs. instantaneous with submission) were the only features of instructional design distinguishing *Conditions* C1 and C2. Weekly one-hour interactive sessions served to distinguish *Condition* C3 from *Condition* C2.

Contrast between Condition C1 and C2.

There were no significant differences in student achievement (*fscore*) or perseverance (*Probability*) of students in the more traditional *Condition* C1 and the WeBWorK *Condition* C2. It should be noted that when we corrected for students' achievement (*fscore* as covariate), students in *Condition* C2 outperformed *Condition* C1 students in perseverance. We speculate that this phenomenon is largely due to the fact that instructors in *Condition* C2 increased students' final grades more than instructors in *Condition* C1. This allowed students in *Condition* C2 to enrol in higher level courses while students in *Condition* C1 were discouraged from doing so because they were more likely to have failed the Calculus I course.

Our original hypothesis that men's motivation and self-efficacy will increase when given the opportunity to use computers to submit their assignments was not supported by the results of this study. Their self-efficacy and motivation was the same in both conditions. Women's motivation did not change either and their self-efficacy remained significantly lower than that of men.

These results contradict some meta-analyses of studies of the effectiveness of CAI which report CAI as being more effective (*e.g.,* Christmann and Badgett, 1997). On the other hand, the result supports the thesis that the positive impact of CAI reported by many studies disappears when there is control for instructional design (Jenks and Springer, 2002). In this study, we not only rigorously controlled the experimental design, but we also meticulously measured students' achievement in terms of *fscore*. We suspect that some of the studies of effectiveness of CAI measures of students' achievement are based on teacher assigned grades, and in this study we have shown that such measures may not be reliable.

Contrast between Condition C3 and Conditions C1 and C2.

Aside from the mode of delivery of assignments and the timing of the delivery of feedback, the *Condition* C3 instructional design also included weekly one-hour long in-class interactive sessions. Women's self-efficacy rose above that of men in this instructional design. It appears that in this learning environment women, in particular, thrived. It is also important to note that students' achievement (*fscore*) in this condition surpassed students' achievement in the two other conditions. It seems that spending 20% of class time providing additional instructional support to students in *Condition* C3 was well worth it, both in terms of motivation of women, and in the performance of both men and women. Students' knowledge of Calculus in *Condition* C3 was superior to that of their peers in *Condition*s C1 and C2. These results support the conclusion of Lowe (2001) that CAI is not a panacea, but rather a tool that can enhance an effective instructional strategy.

Students in *Condition* C3 were also more likely to submit assignments and their solutions of assigned problems were more frequently correct. In other words, the learning environment in *Condition* C3 stimulated students to exert more effort than that of students in the two other conditions. Although there were no significant differences in students' perceptions of the learning environment across the three conditions, the instructors, during post-experiment interviews, reported different student behaviours. All instructors from *Condition* C3 brought up the fact that students frequently e-mailed them questions about assignments. On the other hand, instructors in *Condition*s C1 and C2 did not recall any increase in help-seeking behaviour by their students. It appears that learning environments in *Conditions* C1 and C2 did not promote student effort to the extent that the learning environment in *Condition* C3 did. One reason why the *Condition* C3 instructional design worked better might be because it included instructional support for students, something that Lowe and Holton (2005) consider essential for successful implementation of CAI. Having markers, or even having a computer system to instantly correct assignments, do not by themselves seem to improve learning.

It is particularly important that students in *Condition* C3 were significantly more likely to

Discussion

pursue mathematics in future. We may speculate that if high school and CEGEP teachers were to use this instructional strategy, then the trend of declining enrollment of social science students in Calculus classes at CEGEP might be reversed. In particular, as we noted before, women in the Social Science Program are less likely to take Calculus courses than men, and *Condition* C3 was particularly helpful for women.

Finally, as anticipated when viewing the final grades, failure rates in *Condition* C3 were significantly lower than *Condition* C1 or *Condition* C2. Actually, failure rates in *Condition* C2 were also lower than those in *Condition* C1, however this difference between conditions *Condition* C1 and *Condition* C2 seems to be an artifact of the tendency of instructors in *Condition* C2 to boost final grades more than their colleagues in the other conditions. We also note that failure rates in *Condition* C1 do not differ from those reported in the network of colleges. Thus, it appears that instructors' effort to situate problems in contexts relevant to budding social scientists, and assigning paper-based marker corrected homework, did not by themselves do much to improve learning or motivation to succeed.

We note that virtually all instructors in this experiment were sufficiently impressed with the instructional design in *Condition* C3 that they now employ it in their classes. This result alone is extraordinary because recommendations coming from educational research usually have little impact on actual teaching in sciences and mathematics (Handelsman, Ebert-May, Beichner, Bruns, Chang, De-Haan, Gentile, Lauffer, Stewart, Tilgham and Wood, 2004; Poellhuber, 2001). Although this design requires schools to have a sufficient number of computer labs with Internet connections, this may not be much of an impediment to implementation because many schools and colleges already have such classrooms, or plan to add them shortly.

We also noted an unexpected result. When we studied the relationship between final grades and students' achievement (*fscore*), we noted that although students knowledge of Calculus was significantly lower in C2, the instructors teaching in this condition raised grades more than instructors teaching in the other two conditions. Instructors were surprised when shown this result and stated that they had not consciously raised marks. Perhaps it is not coincidental that the largest boost of final grades happened in the weakest classes. This may also be related to a phenomenon commonly referred to as "grade inflation". If this result can be replicated, it may explain why average grades rise, despite instructors continually complaining that, if anything, their students seem increasingly less well prepared.

Discussion

In the course of this study, we developed measures of students' knowledge of algebra and students' knowledge of functions, both of which were also validated. It was shown that they reliably measure students' knowledge. They are not diagnostic tools that predict students' performance in subsequent courses. This is because students' performance depends on many variables (*e.g.,* learning environment, students' motivation and students' self-efficacy). On the other hand, these tools could help instructors to identify students' weakness in prior knowledge, and adjust their instruction to help students overcome those weaknesses.

Limitations

The results of this study suggest that interactive sessions may enhance the impact of WeBWorK on student learning, but in this study we cannot disentangle the differential impact of WeBWorK and of interactive sessions, since we did not conduct a full 2x2 design (*i.e.*, there were no sections with paper based assignments as well as interactive class sessions working on the assignments). Furthermore, none of the data we collected can explain precisely how the learning environment in *Condition* C3 promoted students' learning. We speculate that it allowed students to ask questions that they would otherwise have been too intimidated to ask. It is also possible that they felt more supported by their instructor, or that there was a heavier emphasis on the importance of doing assignments in C3, by virtue of spending class time on them. The interactive sessions also provided an environment where collaboration with peers was easily initiated and frequently employed by students. More research is needed to clarify the exact mechanisms involved.

Conclusions

VIII. Conclusions

When we began thinking about this project, we were intrigued by complaints from mathematics instructors. They claimed that social science Calculus students are an amotivated group in contrast to science Calculus students. Instructors also claimed that social science students are ill prepared for the study of Calculus. While we have some agreement with the latter claim, given the disjunction between the high school and CEGEP curriculum, we were intrigued by the first claim because social science Calculus students are actually students who chose to enrol in Calculus classes of their own volition, unlike their science peers for whom Calculus is compulsory. That is, most social science students are not obliged to enrol in mathematics courses to satisfy the requirements of their program, where science students have no such choice. Consequently, our initial research objective was to investigate students' motivation, observing and measuring how it changes during Calculus instruction. As we searched the literature concerning this problem, we discovered three important facts: 1. social science Calculus enrolment has steadily declined over the past fifteen years; 2. during the same period, failure rates steadily climbed; 3. failure rates were particularly high amongst men. In view of the fact that working social scientists increasingly depend on sophisticated mathematical skills, these trends appeared worrisome and worthy of investigation. Hence, our research objectives expanded. While controlling for prior knowledge and motivation, our goal in this study became one of evaluating three instructional designs which might reverse the distressing trends outlined above. The research questions in this study then can be stated as follows:

1. Are students ill prepared to study Calculus, and if so, does their lack of preparation have a significant impact on their learning in Calculus?
2. Which, if any, of the three tested experimental conditions (instructional designs) is apt to reverse the trend of increasing failure rates?
3. Are there gender differences in the impact of these three conditions on student achievement and perseverance?
4. Are there gender differences in the impact of these three conditions on student motivation and self-efficacy?

To answer the first question, we developed two measures. The first measures students' algebraic skills (see Appendix A). Our results indicate that instructors' claim that students are ill prepared, has merit. On average, there is a 23% probability that a student in our sample knows

Conclusions

the algebraic operations that most Calculus instructors consider essential to success. Given that reformers of high school curriculum chose to de-emphasize algebra, this result may not be too surprising, but nevertheless it is a problem. The second measure assessed understanding of functions (see Appendix A). Students' success on this measure was even worse than the results on the algebra test. A student who was enrolled in either of two experimental conditions, *Condition* C1 or *Condition* C3, had on average a 20% probability of understanding the concept of functions at levels that Calculus instructors consider essential to success in Calculus courses. A student in *Condition* C2 had on average a 4% probability of understanding this concept. It should be noted that in the course of this study, both measures, Knowledge of Algebra and Knowledge of Functions, were found to have intrinsic and extrinsic validity. This alarmingly low result should worry reformers of high school curriculum since students' conceptual understanding was the aim of the reform.

The answer to the follow-up question - "Does this lack of preparation have a negative impact on student achievement?" - is, in a word, No. The results show that neither students' prior knowledge of algebra nor their prior knowledge of functions had a significant impact on students' achievement and perseverance. In fact, the results show that students' *High School Math Performance* significantly affected students' achievement in Calculus and their perseverance. Note, that students high school performance is indicative of not only their knowledge, but also of their self-regulatory skills. We suspect that high performers in our sample were students who monitored their performance and successfully compensated for the lack of prior skills. On the other hand, the low performers probably did not have the study skills to do so. Mathematics high-school marks encompass not only actual subject knowledge, but also the ability to learn efficiently, an ability which is used profitably by the high-performing students again at the college level. This could explain why achievement was affected by the high school mathematics performance and not by prior knowledge of algebra or functions. In addition, we speculate that most instructors used instructional strategies that helped their students to compensate for the lack of prior knowledge. This speculation is supported by results which show that students' knowledge of algebra and knowledge of functions grew significantly during the course. In conclusion, the lack of prior knowledge and skills may not be the sole cause of escalating failure rates in Calculus courses in this population.

To answer the second research question, the results of this study indicate that we can reverse the downward trend in enrollment and diminish failure rates in mathematics courses at the

Conclusions

CEGEP level if we promote implementation of instructional designs similar to C3, both in CEGEP and secondary schools. When combined with in-class interactive sessions, this form of CAI (WeBWorK assignments) substantially improves student learning and the likelihood of continuing with mathematics studies. In addition, the results show that students' effort in this condition was significantly higher than the effort of their peers in the two other experimental conditions. While delivering and grading assignments via a computer is an efficient alternative to employing human markers, this research shows that providing feedback via WeBWorK alone is not enough to improve students' achievement and perseverance or to promote larger effort on their part.

As indicated by the results of this experiment, mathematics instructors, virtually all of whom firmly believe in the old maxim "practice makes perfect", may be eager to implement a C3 design across the network of colleges, if given the chance. We speculate that high school mathematics instructors may equally endorse this instructional strategy, and use it quite happily in their classes. We anticipate that the consequence of widespread adoption of such a strategy would be increases in enrollment in mathematics, and lower failure rates in Calculus, allowing more social science students across the CEGEP network to successfully meet their career goals. This strategy entails a small startup cost in terms of equipment (one Linux server can service multiple schools), and a small operating cost for technical support for instructors. On the other hand, the human cost, and cost to society, is likely to be much larger if we do not solve the problem and allow the vast majority of a generation of students to graduate from CEGEP without the mathematical prerequisites for so many career options.

We have also, by accident, discovered a worrisome trend in the course of this study. Instructors pushed up the grades of their students and the average grade increase was significantly higher in the least performing group of students. Our sample of instructors was small, and their grading practices may have been affected by their participation in the study. Thus, this result may be spurious, and not indicative of general grading practices amongst mathematics instructors at large. However, it is clear that the final grades in the courses under study did not accurately reflect students' mastery of Calculus concepts. If this result is indicative of general trends in grading, then the school and college administrators who monitor students' success via grades, should be worried. This result necessitates further investigation.

Conclusions

The answer to the third research question - "Are there gender differences in students' achievement and perseverance?" - is affirmative. Women significantly outperformed men in all three conditions and they were also significantly more likely to pursue further studies in mathematics than their male peers. Unfortunately, the male population in our sample was too small to reliably address the issue of impact of working with computers on male students. It appears that working with computers with additional instructional support in *Condition* C3 enhanced their achievement. Given the fact that their achievement is significantly lower than that of women in every condition, further research into their lower academic success is necessary. It appears to us that qualitative research investigating male motivation and/or interest in Calculus may be needed before a successful instructional strategy can be designed to reverse the current trends of failure amongst men.

We have also studied whether the three *Conditions* had a differential impact on students' motivation and self-efficacy. We have found that it did not when prior motivation was controlled for. On the other hand, we have found that self-efficacy of women in classes that included interactive sessions was higher than the self-efficacy of men. It is possible that this effect is due to the fact that women in this *Condition* significantly outperformed men. That is, it could be that the self-efficacy of women rose because they experienced or witnessed success. It is also possible that additional feedback they obtained from their peers and the instructors during the interactive sessions contributed to their increased beliefs about their competence and offset their traditional discomfort when computers are integrated in classes (Butler, 2001). Unfortunately, we cannot disentangle these two possible sources of changes in self-efficacy in this study, because we did not collect data that would allow us to distinguish between the possibilities. It will be most interesting to continue to study this issue because the traditionally low self-efficacy of women in mathematics has long been associated with their low perseverance.

It is important to note that the failure rate (26%) of students in *Condition* C3 was well below the failure rates in traditional social science classes, which hover around 40%. Even though these students had serious gaps in their algebra skills and their understanding of functions when the course began, they and their instructors were able to compensate for this lack of knowledge better than the students and the instructors in *Conditions* C1 and C2. In addition, we need to point out that the content of the course was very traditional. We did not heed to advice of researchers in mathematics education who believe that the content of the Calculus course needs to be reduced. In the context of this study, the content remained intact. The only change in

Conclusions

content consisted of modifying the settings of some problems so that they were drawn from real situations encountered in business environments and sociological studies. However, we did not investigate how the content of this course has changed over the past few years. It is possible that the current content already has changed to compensate for decreases in student prior knowledge of algebra and functions. If so, this would explain why the alarmingly low prior knowledge scores did not have a significant impact on student achievement. The fact that instructors inflated grades may be just a second mechanism they employ to compensate for gaps in students' prior knowledge.

We have identified a successful instructional strategy that was shown to promote achievement and perseverance of students in mathematics courses. Although, it has only been tested amongst CEGEP social science students, there is nothing specific about this population or strategy that would indicate that this strategy would fail to impact similarly on high school students or CEGEP science students. A modest investment in technology could improve the performance of Quebec students vis-a-vis their competitors in other industrialised countries. This in itself is the most important result of this research.

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Appendix A - Algebra and Functions Tests

Algebra Skills Test

Print Name: ____________________ Student Number: ______________

Instructions: In all questions below, either circle the correct answer (there is always only one correct answer), or fill in the blank(s) as instructed.

Problem 1. Which of the answers below is a valid simplification of the following expression:

$$\frac{3}{x-3}+\frac{2x}{x^2-9}$$

A) $\frac{3+2x}{(x-3)+(x^2-9)}=\frac{2x+3}{x^2+x-12}$ B) $\frac{3(2x)}{(x-3)(x^2-9)}=\frac{6x}{x^3-3x^2-9x+27}$

C) $\frac{3(x^2-9)+2x(x-3)}{(x-3)^2+(x^2-9)^2}=\frac{5x^2-6x-27}{x^4-17x^2-6x+90}$

D) $\frac{3(x+3)+2x}{(x^2-9)}=\frac{5x+9}{x^2-9}$ E) $\frac{3-2x}{(x-3)-(x^2-9)}=\frac{3-2x}{-x^2+x+6}$

Problem 2. In the space provided below enter the correct solution for x to the equation given below:

$$\frac{1}{\frac{x}{1+x}}=2$$

$x =$ ______

Problem 3. Which of the answers below is a valid rationalization of the following expression:

$$\frac{1}{\sqrt{2}+\sqrt{3}}$$

A) $\sqrt{5}$ B) $\sqrt{2}-\sqrt{3}$ C) $-\sqrt{2}-\sqrt{3}$ D) $-\sqrt{2}+\sqrt{3}$ E) $\sqrt{2}+\sqrt{3}$

Problem 4. In the space provided below enter a valid solution for x to the equation given below:

$$\sqrt[3]{\sqrt[2]{x}}=2$$

$x =$ ______

Problem 5. Which of the answers below is a valid expansion of the following expression:

$$(x-y)^2$$

A) x^2-y^2 B) $x^2-2xy-y^2$ C) $x^2+2xy+y^2$ D) x^2-xy+y^2 E) $x^2-2xy+y^2$

Appendix A - Algebra and Functions Tests

Problem 6. Using the spaces provided below, enter the two solutions of the following quadratic equation:

$$x^2 - 4x + 3 = 0$$

smaller solution = ________ bigger solution = ________

Problem 7. Possible factors of $2x^2 - x$ are: I) $2x$ II) x III) $2x - 1$. Which of the five following options is correct?

A) All of I, II and III are factors **B)** None of I, II, and III are factors **C)** I and II are factors

D) I and III are factors **E)** II and III are factors

Problem 8. Use the blank below to enter the value of x which is the solution for the following equation:

$$-17x + 9 = 2 + (1 - 2x)$$

x = ________

Problem 9. Use the blanks below to enter your solutions to the following equation:

$$x - \frac{6}{x} = 5$$

smaller solution = ________ bigger solution = ________

Problem 10. Solve the following inequality

$$1 < 5 - 2x < 11$$

Answer: ____ $< x <$ ____

Appendix A - Algebra and Functions Tests

Functions Skills Test

Print Name: ____________________ Student Number: ______________

Instructions: In all questions below, either circle the correct answers (there might be more than one correct answer), or fill in the blank(s) as instructed.

Problem 1. Check the box next to EACH table which represents a function $y = f(x)$.

☐

x	-2	-1	1	1	3	5
y	6	2	3	-1	5	5

☐

x	-2.5	-2.4	-1.6	1.1	1.3	1.4
y	6	2	3	-1	5	5

☐

x	0	-2	1	-1	3	2
y	2	2	3	-1	5	1

☐

x	-2	-1	0	0	1	1
y	6	2	3	-1	5	7

☐

x	-4	-2	-1	1	2	3
y	2	2	3	3	2	2

Appendix A - Algebra and Functions Tests

Problem 2. Some values of the two functions $f(x)$ and $g(x)$ are listed in the two tables below.

x	-5	-2	-1	1	3
$f(x)$	6	2	3	-1	5

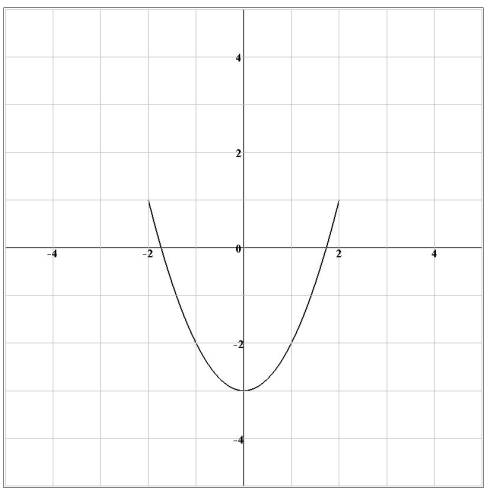
x	-5	-2	-1	1	3
$g(x)$	-2	0	-1	4	-2

Use values from the tables above to compute the following requested values, and then write them in the adjacent blanks. Write ND if the value is undefined.

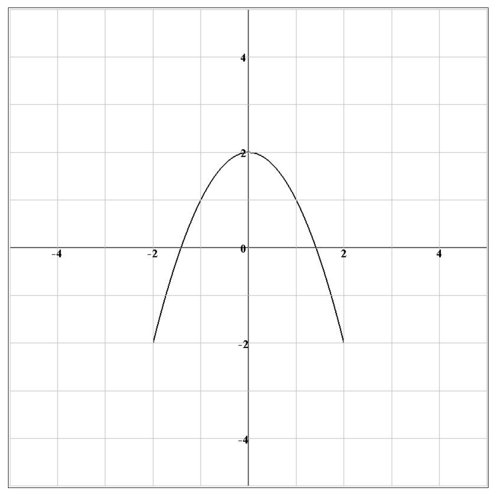
$(f + g)(-1)$ = _____ $(f - g)(3)$ = _____

$(fg)(-5)$ = _____ $(f/g)(-2)$ = _____

Problem 3. Graphs of the two functions $f(x)$ and $g(x)$ are shown below.



$f(x)$



$g(x)$

Using the graphs above determine the following requested values, and then write them in the adjacent blanks. Write ND if the value is undefined.

$(f \circ g)(0)$ = _____ $(f \circ f)(0)$ = _____

$(g \circ f)(0)$ = _____ $(g \circ g)(0)$ = _____

Appendix A - Algebra and Functions Tests

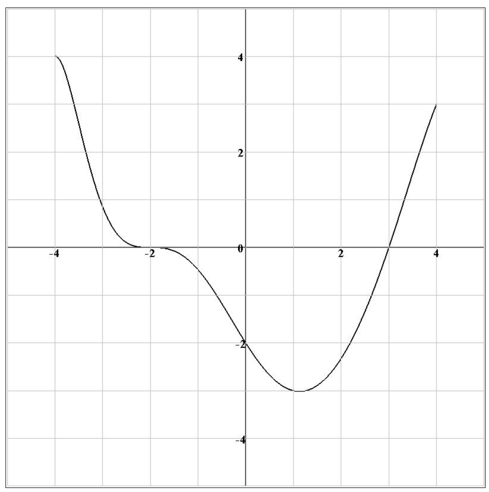
Problem 4. Given the function

$$f(x) = \frac{1}{x-2}$$

in the blank below write the formula for $f^{-1}(x)$.

$f^{-1}(x) =$ _____

Problem 5. A graph of the function $f(x)$ is shown below.



$f(x)$

Use the above graph and compute the average rate of change of $f(x)$ for each of the following requested intervals, and then write the rate into the adjacent blank.

Rate of change on [-4,0] = _____,

Rate of change on [1,4] = _____

Appendix A - Algebra and Functions Tests

Problem 6. Each of the three tables below represents a function. Enter the slope for each function which is linear using the corresponding blank. Leave empty any blank corresponding to a non-linear function.

x	-5	-3	-1	1	3
$f(x)$	6	6	-2	-6	-10

Slope = _____

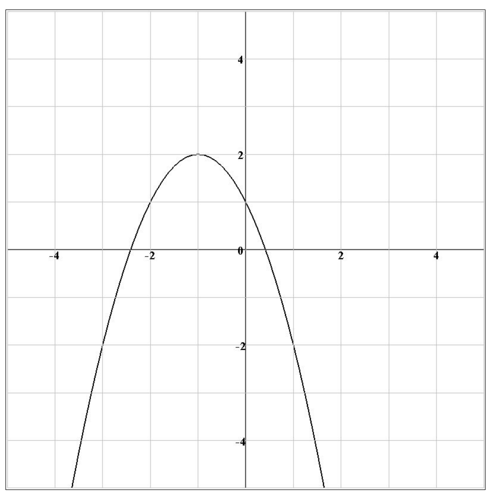
x	-5	-3	-1	1	3
$g(x)$	-2	0	-1	4	-2

Slope = _____

x	2	2.1	2.2	2.3	2.4
$h(x)$	-2	-1	0	1	2

Slope = _____

Problem 7. The graph below represents a quadratic function of the form $f(x) = -(x + a)^2 + b$. From the graph determine a and b.

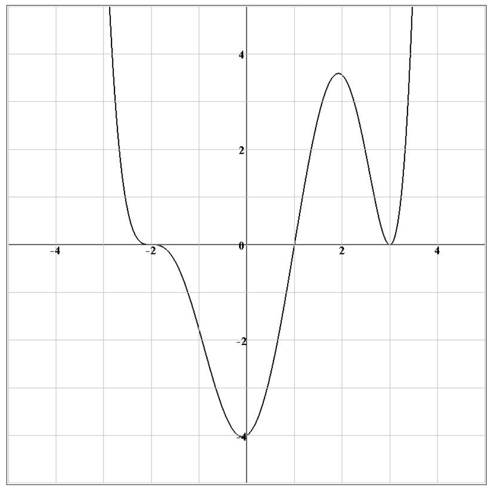


$f(x)$

a = _____ b = _____

Appendix A - Algebra and Functions Tests

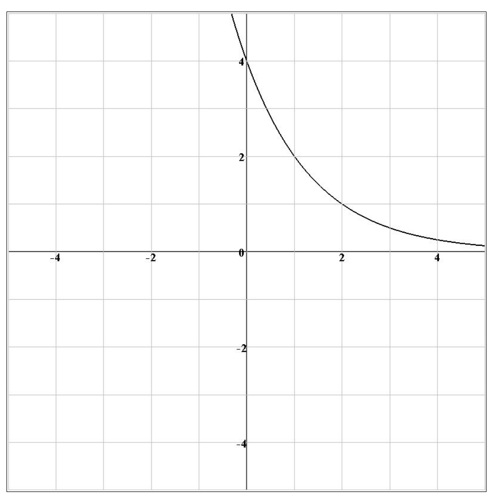
Problem 8. The graph below represents a polynomial function of the form $f(x) = A(x - b)(x - c)^2(x - d)^3$. From the graph determine b, c, d.



$f(x)$

b = _____ c = _____ d = _____

Problem 9. The graph below represents an exponential function of the form $f(x) = a(b^x)$. From the graph determine a and b.



$f(x)$

a = _____ b = _____

Appendix A - Algebra and Functions Tests

Problem 10. Jamie is practising for a skateboard competition at the neighbourhood park. The ramp is in the shape of a sinusoidal function. The grey part of the graph below represents the height, $f(x)$, of the ramp as a function of the horizontal distance x.

Which of the following functions correctly represents the shape of the ramp seen in the graph? Check the appropriate box.

☐ $4\sin(x)$

☐ $2\cos(x)+2$

☐ $2\cos\left(\frac{\pi}{6}x\right)+2$

☐ $4\cos(6\pi x)$

☐ $4\cos\left(\frac{\pi}{6}x\right)$

Appendix B - Common Questions from Term Tests

From Term Test 1:

1. (a) Jack has been doing some overtime work and his employer just rewarded him with a \$15,000 bonus. Jack is engaged to get married so this extra money will come in handy because his fiancee Jenny wants a big wedding and nobody else is going to help pay for the wedding. Jack decides not to tell Jenny because he is worried that she tends to spend money. Instead he will lock the money into a GIC to help it grow, and avoid spending it. His bank is currently offering 6%, compounded continuously for a GIC of two years or longer. If the wedding that Jenny wants will cost \$20,000, how long will they have to wait?

   (b) Unbeknownst to Jack, Jenny has also just earned a big bonus at work, also \$15,000. Jenny is afraid that Jack will want to spend it all so she too decides not to tell, but to invest it so that she can have that big wedding that she really wants that will cost \$20,000. She is willing to wait at most three years to get married. If the bank is offering GIC's that are compounded daily, what is the lowest interest rate that will get her the \$20,000 for the wedding in three years?

Solutions:

(a) The phrase "compounded continuously" tells us that the formula here is $V(t) = Pe^{it}$ (E#1) is the appropriate formula to use for this problem. The phrase "is currently offering 6%" tells us that $i = 0.06$. The phrases "a \$15,000 bonus" and "lock the money" tell us that $P = 15000$. The last sentence tells us both that $V(t) = 20000$ and t is what we need to solve for.

Substituting in all the given information transforms E#1 into $20000 = 15000e^{0.06t}$ (E#2)

Dividing both sides of E#2 by 15000, and simplifying fractions, yields $(4/3) = e^{0.06t}$ (E#3)

Taking ln of both sides of E#3, and using the fact that ln( ) and $e^{( )}$ are inverses, yields $\ln\left(\frac{4}{3}\right) = 0.06t$ (E#4)

Dividing both sides of E#4 by 0.06, and then evaluating with a calculator, we obtain: $t = \frac{\ln\left(\frac{4}{3}\right)}{0.06} \doteq 4.7947$

That is, Jack will have to wait almost 5 years before this investment will bring in the \$20,000 that he needs to pay for the wedding that Jenny wants.

(b) The phrase "compounded daily" tells us that the formula here is $V(t) = P\left(1+\frac{i}{n}\right)^{nt}$ (E#1) is the appropriate formula to use for this problem and that $n = 365$. The phrases "a big bonus at work, also \$15,000" and "to invest it" tell us that $P = 15000$. The phrase "will cost \$20,000" tells us that $V(t) = 20000$. The phrase "wait at most three years" tells us that $t = 3$. The last sentence tells us what we wish to solve for is i.

Substituting in all the given information transforms E#1 into $20000 = 15000\left(1+\frac{i}{365}\right)^{365\times 3}$ (E#2)

Dividing both sides of E#2 by 15000, and multiplying out the exponent, yields $\frac{4}{3} = \left(1+\frac{i}{365}\right)^{1095}$ (E#3)

Taking 1095 root (or 1/1095 power) of both sides of E#3, yields $\left(\frac{4}{3}\right)^{1/1095} = \left(1+\frac{i}{365}\right)$ (E#4)

Subtracting 1 from both sides, then multiplying both sides of E#4 by 365, and evaluating with a calculator, we obtain:

$$i = 365\left(\left(\frac{4}{3}\right)^{1/1095} - 1\right) \doteq 0.095907$$

That is, she will need an interest rate of about 9.6% so that she will have the \$20,000 to pay for her wedding in three years time.

Appendix B - Common Questions from Term Tests

From Term Test 2:

1. For each of the following limits:
   a. if the limit of the function exists, determine the value of the limit; if not, explain why not.
   b. draw a small sketch of the part of the function nearby the value that x approaches to illustrate what your limit result has told you.

   i) $\lim_{x\to 2}\frac{x^2-4}{(x-2)^2}$ ii) $\lim_{x\to -2}\frac{x^2-4}{(x-2)^2}$ iii) $\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}$

Solution:

i) Undefined or DNE. The half-limits tell us that this function has an infinite discontinuity at $x = 2$, *i.e.*, a graph of this function has a vertical asymptote $x = 2$, and on the left of that vertical line a graph would head downward towards $-\infty$, but on the right a graph would head upward towards ∞.

ii) This function is continuous at $x = -2$, and $f(-2) = 0$.

iii) Although $f(2)$ is undefined, the limit exists and is 0. This means that the function has a removable discontinuity at $x = 2$, *i.e.*, a graph will have a missing point at (2,0).

2. Given $f(x) = 2x^2 - 3x + 5$:
   a. Use the Newton's Quotient definition of the derivative, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, to prove that

   $$f'(x)=\frac{df(x)}{dx}=4x-3$$

(Note that no marks will be given for using the rules of differentiation in this example.)

   b. Use the derivative verified above to determine an equation for the line tangent to f at $x = 2$

$$\frac{y-y_0}{x-x_0}=m\Rightarrow\frac{y-7}{x-2}=5\overset{\text{(optional)}}{\Leftrightarrow}y-7=5(x-2)\Leftrightarrow y=5x-3$$

3. When electric blenders are sold for p dollars apiece, local consumers will buy $Q(p)=\frac{8000}{p}$ blenders per month. It is estimated that t months from now the price of these blenders will be p, where $p(t) = 0.04t^2 + 15$ dollars.
   a. Write Q as a function of t.
   b. Compute the quantity Q demanded at $t = 25$ months.
   c. Compute the rate at which the monthly quantity of blenders demanded will be changing with respect to time 25 months from now.
   d. Will the quantity Q of blenders demanded be increasing or decreasing in 25 months?

Solutions:

a. $Q(p)=\frac{8000}{p}\Rightarrow Q(p(t))=\frac{8000}{p(t)}=\frac{8000}{0.04t^2+15}$

b. $Q(25)=\frac{8000}{0.04(25)^2+15}=200$

c.

$$Q'(t)=\frac{dQ}{dt}=\frac{d\left(\frac{8000}{0.04t^2+15}\right)}{dt}=\frac{d\left(8000\left(0.04t^2+15\right)^{-1}\right)}{dt}=-\frac{640t}{\left(0.04t^2+15\right)^2}$$

$$\Rightarrow Q'(25)=\left.\frac{dQ}{dt}\right|_{t=25}=-\frac{640(25)}{\left(0.04(25)^2+15\right)^2}=-\frac{\overset{10}{\cancel{16000}}}{\cancel{(1600)}}=-10$$

d. The quantity of blenders demanded will be decreasing in 25 months.

Appendix B - Common Questions from Term Tests

From Term Test 3:

1. For each of the functions defined below use the rules and techniques of differentiation learned in class to compute the requested derivative:

(6) (a) Given $f(x) = (3x^{10} - 4x^{-3})\sin(x)$, determine $f'(x) = \frac{df(x)}{dx}$

Solution:

$$
\begin{aligned}
f'(x) = \frac{df(x)}{dx} &= \frac{d\left(\left(3x^{10}-4x^{-3}\right)\sin(x)\right)}{dx} \\
&= \left(\frac{d3x^{10}-4x^{-3}}{dx}\right)\sin(x) + \left(3x^{10}-4x^{-3}\right)\left(\frac{d\sin(x)}{dx}\right) \quad \text{(Product Rule)} \\
&= \left(\frac{d3x^{10}}{dx} - \frac{d4x^{-3}}{dx}\right)\sin(x) + \left(3x^{10}-4x^{-3}\right)\left(\cos(x)\right) \quad \text{(Difference \& Sine Rules)} \\
&= \left(3\frac{dx^{10}}{dx} - 4\frac{dx^{-3}}{dx}\right)\sin(x) + \left(3x^{10}-4x^{-3}\right)\cos(x) \quad \text{(Constant Multiple Rule)} \\
&= \left(3\left(10x^{9}\right) - 4\left(-3x^{-4}\right)\right)\sin(x) + \left(3x^{10}-4x^{-3}\right)\cos(x) \quad \text{(Power Rule)} \\
&= \left(30x^{9} + 12x^{-4}\right)\sin(x) + \left(3x^{10}-4x^{-3}\right)\cos(x) \quad \text{(Arithmetic/Algebra/Functions Cleanup)}
\end{aligned}
$$

(6) (b) Given $g(y) = \frac{e^y}{\cos(y)}$, determine $g'(y) = \frac{dg(y)}{dy}$

Solution:

$$
\begin{aligned}
g'(y) = \frac{dg(y)}{dy} &= \frac{d\left(\frac{e^y}{\cos(y)}\right)}{dy} \\
&= \frac{\left(\frac{de^y}{dy}\right)\cos(y) - e^y\left(\frac{d\cos(y)}{dy}\right)}{\left(\cos(y)\right)^2} \quad \text{(Quotient Rule)} \\
&= \frac{\left(e^y\right)\cos(y) - e^y\left(-\sin(y)\right)}{\cos^2(y)} \quad \text{(e \& Cosine Rules)} \\
&= e^y\left(\frac{\cos(y)+\sin(y)}{\cos^2(y)}\right) \quad \text{(Arithmetic/Algebra/Functions Cleanup)}
\end{aligned}
$$

Appendix B - Common Questions from Term Tests

(6) (c) Given $h(t) = t^{(3t-2)}$, determine $h'(t) = \dfrac{dh(t)}{dt}$

Solution:
We note immediately that the function *h* is of the form (*variable expression #1*)$^{(variable\ expression\ \#2)}$, so we must use the technique of logarithmic differentiation.

$$h(t) = t^{(3t-2)} \Rightarrow \ln\left(h(t)\right) = \ln\left(t^{(3t-2)}\right) = (3t-2)\ln(t) \quad \textbf{(Log. Diff. Technique)}$$

$$\frac{d\ln\left(h(t)\right)}{dt} = \frac{d(3t-2)\ln(t)}{dt} \quad \textbf{(Log. Diff. Technique)}$$

$$\Leftrightarrow \frac{d\ln\left(h(t)\right)}{dh(t)} \times \frac{dh(t)}{dt} = \left(\frac{d(3t-2)}{dt}\right)\ln(t) + (3t-2)\left(\frac{d\ln(t)}{dt}\right) \quad \textbf{(Chain \& Product Rules)}$$

$$\Leftrightarrow \frac{1}{h(t)} \times \frac{dh(t)}{dt} = \left(\frac{d3t}{dt} - \frac{d2}{dt}\right)\ln(t) + (3t-2)\left(\frac{1}{t}\right) \quad \textbf{(ln \& Difference Rules)}$$

$$\Leftrightarrow \frac{1}{h(t)} \times \frac{dh(t)}{dt} = \left(3\frac{dt}{dt} - 0\right)\ln(t) + (3t-2)\left(\frac{1}{t}\right) \quad \textbf{(Constant Multiple \& Constant Rules)}$$

$$\Leftrightarrow \frac{1}{h(t)} \times \frac{dh(t)}{dt} = \left(3(1)\right)\ln(t) + (3t-2)\left(\frac{1}{t}\right) \quad \textbf{(Identity Rule)}$$

$$\Leftrightarrow \frac{dh(t)}{dt} = \left(3\ln(t) + (3t-2)\left(\frac{1}{t}\right)\right)h(t) \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow \frac{dh(t)}{dt} = \left(3\ln(t) + (3t-2)\left(\frac{1}{t}\right)\right)t^{(3t-2)} \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

(6) 2. Given $x^3y^2 + 3y^2 = 5x^3 - y^3$, determine $y' = \dfrac{dy}{dx}$ and an equation for the line tangent to the given curve at the point (1, 1).

Solutions:
We note that this example involves *y* implicitly defined as a function of *x*, and hence we use implicit differentiation.

Appendix B - Common Questions from Term Tests

$$x^3y^2+3y^2=5x^3-y^3 \Rightarrow \frac{d\left(x^3y^2+3y^2\right)}{dx}=\frac{d\left(5x^3-y^3\right)}{dx} \quad \text{(Implicit Differentiation)}$$

$$\Leftrightarrow \frac{d\left(x^3y^2\right)}{dx}+\frac{d\left(3y^2\right)}{dx}=\frac{d\left(5x^3\right)}{dx}-\frac{d\left(y^3\right)}{dx} \quad \text{(Sum \& Difference Rules)}$$

$$\Leftrightarrow \left(\frac{dx^3}{dx}y^2+x^3\frac{dy^2}{dx}\right)+3\frac{dy^2}{dx}=5\frac{dx^3}{dx}-\frac{dy^3}{dy}\times\frac{dy}{dx} \quad \text{(Product, Constant Multiple \& Chain Rules)}$$

$$\Leftrightarrow \left(\left(3x^2\right)y^2+x^3\frac{dy^2}{dy}\times\frac{dy}{dx}\right)+3\frac{dy^2}{dy}\times\frac{dy}{dx}=5\left(3x^2\right)-\left(3y^2\right)\times\frac{dy}{dx} \quad \text{(Power \& Chain Rules)}$$

$$\Leftrightarrow \left(3x^2y^2+x^3(2y)\frac{dy}{dx}\right)+3(2y)\frac{dy}{dx}=15x^2-3y^2\frac{dy}{dx} \quad \text{(Power Rule)}$$

$$\Leftrightarrow 3x^2y^2+2x^3y\frac{dy}{dx}+6y\frac{dy}{dx}=15x^2-3y^2\frac{dy}{dx} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow 2x^3y\frac{dy}{dx}+6y\frac{dy}{dx}+3y^2\frac{dy}{dx}=-3x^2y^2+15x^2 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow \left(2x^3y+6y+3y^2\right)\frac{dy}{dx}=-3x^2y^2+15x^2 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow \frac{dy}{dx}=\frac{-3x^2y^2+15x^2}{2x^3y+6y+3y^2} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

To determine the equation of a tangent line we typically need two pieces of information: the point of tangency; the slope of the curve (and hence of the tangent line) at the point of tangency. Clearly we have been given the point (1,1) which satisfies the equation that defines the function because $x^3y^2+3y^2=5x^3-y^3 \Rightarrow (1)^3(1)^2+3(1)^2 \,?\, 5(1)^3-(1)^3 \Leftrightarrow 1+3\,?\,5-1 \Leftrightarrow 4=4$. In the calculation above we have solved for the derivative, $\frac{dy}{dx}=\frac{-3x^2y^2+15x^2}{2x^3y+6y+3y^2}$, however we now must compute the value of that derivative at the point (1,1): $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=\frac{-3(1)^2(1)^2+15(1)^2}{2(1)^3(1)+6(1)+3(1)^2}=\frac{-3+15}{2+6+3}=\frac{12}{11}$. Now, using the point-slope form of the equation of a straight line we compute an equation for the tangent line to the curve at the point (1,1):

$$\frac{y-1}{x-1}=\frac{12}{11}, \text{ or optionally, } y-1=\frac{12}{11}(x-1) \Leftrightarrow y=\frac{12}{11}x-\frac{1}{11}$$

(8) 3. Microsoft has noticed that with the big publicity surrounding the introduction of the Sony PlayStation 3, sales of its Xbox 360 have begun to taper off. One of their bright young new hires in marketing has watched previous comparable situations and has built a model for demand, *D*, as a function of price, *p*: $D(p)=1-\frac{\ln(p)}{p}$ and that $p(12)=4$ and $\left.\frac{dp}{dt}\right|_{t=12}=2$.

Determine $\left.\frac{dD}{dt}\right|_{t=12}$. Would the demand be increasing or decreasing after 12 weeks?

Solution:

Step 0: We read the problem over at least twice.

We note that we have been given a rate, $\left.\frac{dp}{dt}\right|_{t=12}=2$, and asked to compute a rate, $\left.\frac{dD}{dt}\right|_{t=12}$, and quite clearly demand *D* and price *p* are related to each other (in fact we have been given the precise relationship, $D(p)=1-\frac{\ln(p)}{p}$), so this is clearly a related rates problem.

Step 1: We draw a diagram to illustrate the problem and to allow us to extract one or more equations relating the variable(s)

Appendix B - Common Questions from Term Tests

whose rate was given to the variable(s) whose rate was requested.
Since we are already given the precise relationship there is no need for a diagram in this problem.
Step 2: From the problem we extract the given rate(s) and the requested rate, and write them in derivative notation.
There is not much to extract or rewrite since the rates were already given in derivative notation.
Step 3: From the diagram we extract one or more equations showing the relationship between variables.
Since we are already given the precise relationship there is no need to "extract" this.
Step 4: Differentiate the equation(s).

$$D(p) = 1 - \frac{\ln(p)}{p} \Rightarrow \frac{dD}{dp} = \frac{d\left(1 - \frac{\ln(p)}{p}\right)}{dp} = \frac{d(1)}{dp} - \frac{d\left(\frac{\ln(p)}{p}\right)}{dp} \text{ (Difference Rule)}$$

$$\frac{dD}{dp} = 0 - \left[\frac{\frac{d\ln(p)}{dp}p - \ln(p)\frac{dp}{dp}}{p^2}\right] \text{ (Constant \& Quotient Rules)}$$

$$= -\left[\frac{\frac{1}{p}p - \ln(p)(1)}{p^2}\right] \text{ (ln and Identity Rules)}$$

$$= -\left[\frac{1 - \ln(p)}{p^2}\right] \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

While this calculation tells us $\frac{dD}{dp}$, the rate that we want to know is $\frac{dD}{dt}$. However, these derivatives are connected by the Chain Rule, that is: $\frac{dD}{dt} = \frac{dD}{dp} \times \frac{dp}{dt}$.

Step 5: After differentiating we substitute in known values and solve for the rate we were asked to determine.
Since we know both of the derivatives on the right hand side of this equation, we now substitute in and compute the requested derivative: $\frac{dD}{dt} = \frac{dD}{dp} \times \frac{dp}{dt} = -\left[\frac{1-\ln(p)}{p^2}\right] \times \frac{dp}{dt} \Rightarrow \left.\frac{dD}{dt}\right|_{t=12} = -\left[\frac{1-\ln(p(12))}{(p(12))^2}\right] \times \left.\frac{dp}{dt}\right|_{t=12} = -\left[\frac{1-\ln(4)}{4^2}\right](2) = \frac{\ln(4)-1}{8} \doteq 0.048$

Step 6: As with all word problems, rephrase the sentence asking the question and deliver the answer as a sentence in English.
We were asked: "Determine $\left.\frac{dD}{dt}\right|_{t=12}$. Would the demand be increasing or decreasing after 12 weeks?"
Thus, our answer is, the rate of change of demand with respect to time, after 12 weeks, would be about 0.048, and since this is positive, demand is in fact increasing.

(16) 4. Given $f(x) = -xe^{\left(-\frac{x}{2}\right)} + 1$, $f'(x) = \left(\frac{1}{2}\right)(x-2)e^{\left(-\frac{x}{2}\right)}$ and $f''(x) = \left(-\frac{1}{4}\right)(x-4)e^{\left(-\frac{x}{2}\right)}$, use methods learned in your Calculus I class this term to determine the existence and location of: vertical and horizontal asymptotes (if any); critical numbers; intervals where the function is increasing/decreasing; local minima and maxima; points of inflection; intervals where the function is concave up/concave down. Then **sketch** a graph of the given function on the grid supplied below.
Solution:

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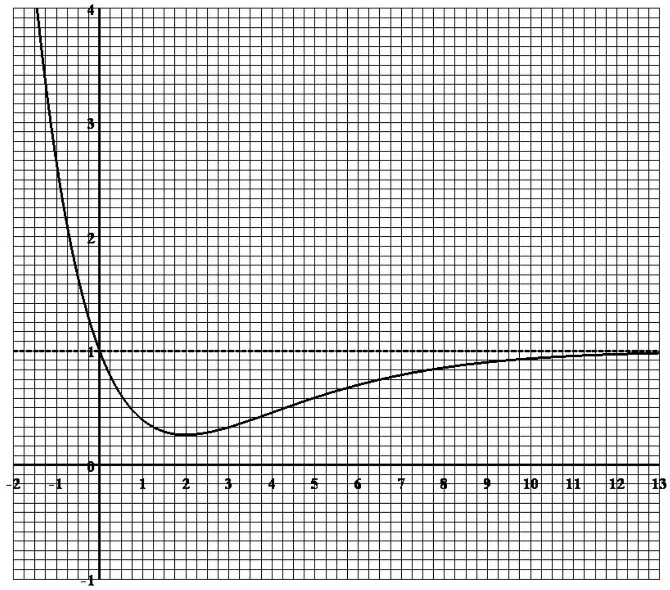


Step 1: What does *f* tell us about a graph of *f* ?
y-intercept: $f(0) = -0(e^{-0}) + 1 = 1$

x-intercepts: $f(x) = 0 \Leftrightarrow -xe^{\left(-\frac{x}{2}\right)} + 1 = 0$, we note that this would be too hard an equation to solve so we do not bother

discontinuities: since $-x$, $e^{-x/2}$ and 1 are all continuous, and $e^{-x/2}$, which really is in the denominator of *f*, is never 0 (exponential decay or growth never touch the *x*-axis), the function *f* has no discontinuities

edge behaviour: we must compute limits, and since exponential functions behave differently at the two edges, two separate limits make sense.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(1 - \frac{x}{e^{x/2}}\right)$$

Now at the left edge of the graph $e^{x/2}$, a growth exponential function, is asymptotic to the *x*-axis, *i.e.*, $\lim_{x\to-\infty}\left(e^{x/2}\right) = 0^{+}$. Thus, in the limit of *f* as *x* approaches $-\infty$, the denominator is approaching 0, making the fraction get large. Of course the numerator is heading towards $-\infty$, which is also making the fraction large in size. Thus, both numerator and denominator are pulling in the same direction. Accounting for signs, we see that this means the function heads towards ∞. That is,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(1 - \frac{x}{e^{x/2}}\right) = \infty$$

Now, for the right edge, in advance we note that $\lim_{x\to\infty}\left(e^{x/2}\right) = \infty$, and we know from early in the course that exponential growth functions such as this "dominate" power functions at the left edge. Thus, when the numerator *x* heads towards ∞, making the fraction larger, and the denominator $e^{x/2}$ heads towards infinity, making the fraction get smaller, the denominator here is more powerful and "wins the battle", pulling the ratio towards 0. Thus, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(1 - \frac{x}{e^{x/2}}\right) = 1 - 0 = 1$. This means that the function *f* has no horizontal asymptote at the left edge, but is asymptotic to the horizontal line $y = 1$ at the right edge. We can even observe that the fraction part of *f* is always positive, so a graph of *f* will approach 1 from underneath the horizontal line (*f* is 1 - small positive number).

Step 2: What does *f* ′ tell us about a graph of *f* ?

The function $f'(x) = \left(\frac{1}{2}\right)(x-2)e^{\left(-\frac{x}{2}\right)} = \frac{(x-2)}{2e^{\left(\frac{x}{2}\right)}}$ is a quotient of a linear function, $(x - 2)$ and an exponential growth function, $2e^{x/2}$.

Since both linear and exponential growth functions are continuous, so *f* ′ is also continuous, except if the denominator is zero. However, exponential growth functions are never zero, so *f* ′ is continuous everywhere. This means that the only critical numbers for *f* would be where $f'(x) = 0$. Clearly this is only true if the numerator is 0, which is at $x = 2$.
We will use $x = 0$ to test the sign of *f* ′ on the interval $(-\infty,2)$, and $x = 3$ to test the sign of *f* ′ on the interval $(2,\infty)$.

$$f'(x) = \frac{(x-2)}{2e^{\left(\frac{x}{2}\right)}} \Rightarrow f'(0) = \frac{-}{+} = - \ \& \ f'(3) = \frac{+}{+} = +$$

Thus, we deduce that *f* is decreasing on $(-\infty,2)$ and increasing on $(2,\infty)$, and that *f* will have a local (and global) minimum at $x = 2$.

Since $x = 2$ is important we calculate $f(2)$: $f(2) = -2e^{\left(-\frac{2}{2}\right)} + 1 = -\frac{2}{e} + 1 \doteq 0.26$

Step 3: What does *f* ″ tell us about a graph of *f* ?

We begin by determining all points where *f* ″ $\left(f''(x) = \left(-\frac{1}{4}\right)(x-4)e^{\left(-\frac{x}{2}\right)} = -\frac{x-4}{4e^{\left(\frac{x}{2}\right)}}\right)$ is either 0 or discontinuous, since according to the Intermediate Value Theorem, these are the only points where *f* ″ might change sign, hence the only points where *f* might change concavity. We note that *f* ″ is a ratio of a linear function and an exponential growth function, both of which are continuous

Appendix B - Common Questions from Term Tests

everywhere, and that the denominator is never 0. Thus, f'' is never discontinuous, and can only be 0 where the numerator is 0, which is at $x = 4$. We will use $x = 0$ to test the sign of f'' on the interval $(-\infty,4)$ and $x = 5$ to test the sign of f'' on the interval $(4,\infty)$.

$$f''(x) = -\frac{x-4}{4e^{\left(\frac{x}{2}\right)}} \Rightarrow f''(0) = -\frac{-}{+} = + \;\&\; f''(5) = -\frac{+}{+} = -$$

Thus, we deduce that f is concave up on $(-\infty,4)$ and concave down on $(4,\infty)$, and that f has a point of inflection at $x = 4$. Since $x = 4$ is important we calculate $f(4)$: $f(4) = -4e^{\left(-\frac{4}{2}\right)} + 1 = -\frac{4}{e^2} + 1 \doteq 0.46$

Having assembled all of the information deduced above in a table we can actually see the shape of the graph. Finally we sketch a graph of f incorporating this information.

			y-int.		*m*		PI		HA
x	-∞		0		2		4		∞
f (*x*)	∞	↘	1 ↘	↘	0.26 →	↗	0.46 ↗	↗	1 →
f ′(*x*)		- - -	- - -	- - -	0	+ + +	+ + +	+ + +	
f ″(*x*)		+ + +	+ + +	+ + +	+ + +	+ + +	0	- - -	
		⌣ ⌣ ⌣	⌣ ⌣ ⌣	⌣ ⌣ ⌣	⌣ ⌣ ⌣	⌣ ⌣ ⌣		⌢ ⌢ ⌢	

Appendix C - Common Final Exam

Common Final Exam:

(6) 1. Evaluate each limit, if it exists. If not, explain why not.

a) $\lim\limits_{x\to-2}\dfrac{3x^2+11x+10}{x^2-4}$ b) $\lim\limits_{x\to-2}\dfrac{3x+5}{(x+2)^2}$

Solutions:

a) The function is rational, hence continuous where defined. Thus our first attempt to calculate the limit is carried out as simple substitution of -2 for *x*.

$$\lim_{x\to-2}\frac{3x^2+11x+10}{x^2-4}=\frac{3(-2)^2+11(-2)+10}{(-2)^2-4}=\frac{12-22+10}{4-4}=\text{“}\frac{0}{0}\text{”}$$

Since substitution failed, *i.e.*, the function is undefined at x = -2, and because substitution gave us $\frac{0}{0}$, we know that

(x - (-2)) = (x + 2) must be a factor in both the numerator and denominator. Thus, our next step is to factor both numerator and denominator, cancel common factor(s), and try substitution again.

$$\lim_{x\to-2}\frac{3x^2+11x+10}{x^2-4}=\lim_{x\to-2}\frac{(3x+5)\,(x+2)}{(x-2)\,(x+2)}=\frac{(3(-2)+5)}{((-2)-2)}=\frac{-1}{-4}=\frac{1}{4}$$

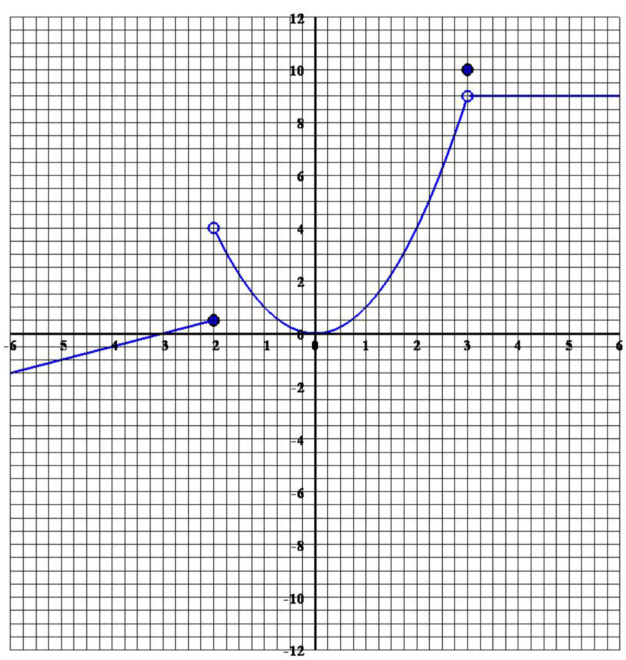
Note that since the limit exists, but the function is undefined at x = -2, the function is commonly said to have a “removable discontinuity” at x = -2, and on a graph of the function this is commonly drawn as a hollow circle at (-2, ¼) indicating that the point is missing from the graph.

b) The function is rational, hence continuous where defined. Thus our first attempt to calculate the limit is carried out as simple substitution of -2 for *x*.

$$\lim_{x\to-2}\frac{3x+5}{(x+2)^2}=\frac{3(-2)+5}{((-2)+2)^2}=\text{“}\frac{-1}{0}\text{”}$$

Since substitution failed, *i.e.*, the function is undefined at x = -2, and because substitution gave us a non-zero number divided by zero we understand that values of the function are getting large in absolute value as x approaches -2. (This is because dividing by a number close to zero, such as 0.1, 0.01, 0.001, etc., is really like multiplying by 10, 100, 1000 etc.) Thus, we can deduce that this function has an infinite discontinuity at x = -2 and a graph of this function has a vertical asymptote at x = -2. Since the denominator in this function is squared, without using half-limits (one-sided limits) we can tell that the denominator approaches 0 through positive numbers. Because the numerator is negative, the ratio’s must all be negative, hence the function is approaching -∞ as x approaches -2.

2. Given $f(x)=\begin{cases}\frac{1}{2}x+\frac{3}{2}, & x\le -2\\ x^2, & -2<x<3\\ 10, & x=3\\ 9, & 3<x\end{cases}$



(2) i. Use the grid below right to sketch a graph of $f(x)$

(6) ii. Determine if $f(x)$ has any discontinuities. If none, explain! If there are discontinuities, for each discontinuity give the x-value of the discontinuity.

Solutions:

b) We can see that *f* is a piecewise defined function, with one piece linear, one quadratic, and two constant. Since all the above function types are continuous everywhere, *f* would be continuous on any interval that involves solely one piece. The only possible discontinuities would be where the function changes from one formula to another, at the “seams”, which are x = -2 and x = 3. On our graph we see a vertical jump at x = -2, hence the function *f* has a jump discontinuity there. On our graph we see a “misplaced point” at x = 3, hence the function *f* has a removable discontinuity there.

Appendix C - Common Final Exam

3. Given $f(x) = 2x^2 - 5x + 3$

(5) i. Use the Newton's Quotient definition of the derivative, $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$, to prove that $f'(x) = 4x - 5$. **<u>Note that no marks will be given for using the rules of differentiation in this example.</u>**

(3) ii. Use the derivative verified above to determine an equation for the tangent line to $f(x)$ at $x = -1$.

Solutions:

a) We do this step by step to minimize the chance of error:

$$f(\ ) = 2(\ )^2 - 5(\ ) + 3$$
$$f(x+h) = 2(x+h)^2 - 5(x+h) + 3 = \cancel{2x^2} + 4xh + 2h^2 \cancel{-5x} - 5h + \cancel{3}$$
$$f(x) = 2x^2 - 5x + 3 = \cancel{2x^2} \cancel{-5x} + \cancel{3}$$
$$f(x+h) - f(x) = 4xh + 2h^2 - 5h = h(4x + 2h - 5)$$
$$\frac{f(x+h)-f(x)}{h} = \frac{\cancel{h}(4x+2h-5)}{\cancel{h}} = (4x + 2h - 5)$$
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(4x+2h-5) = 4x + 2(0) - 5 = 4x - 5$$

b) To determine the equation of this tangent line we need i) the point of tangency and ii) the slope of this tangent line. We are given the x-coordinate of the point of tangency, $x = -1$. Just as in high school, we compute the y-coordinate by substituting this value into the formula for $f(x)$. That is, $f(-1) = 2(-1)^2 - 5(-1) + 3 = 2 + 5 + 3 = 10$, so the point of tangency is $(-1, 10)$. To obtain the slope of this tangent line we substitute $x = -1$ into the formula for $f'(x)$. That is, $f'(-1) = 4(-1) - 5 = -9$. Now, using the point-slope form of the equation for a straight line we obtain:

$$\frac{y-10}{x-(-1)} = -9 \Leftrightarrow \frac{y-10}{x+1} = -9 \overset{\text{optional}}{\Leftrightarrow} y - 10 = -9(x+1) \Leftrightarrow y = -9x + 1$$

Appendix C - Common Final Exam

4. Determine the derivative of each of the following functions:

(7) i. $f(x) = \left(x^3 - 3x^2 + 7x + 5\right)\left(x^3 - 4\right)$

Solution:

$$f'(x) = \frac{df}{dx} = \frac{d\left(\left(x^3 - 3x^2 + 7x + 5\right)\left(x^3 - 4\right)\right)}{dx}$$

(Product Rule)

$$= \left(\frac{d\left(x^3 - 3x^2 + 7x + 5\right)}{dx}\right)\left(x^3 - 4\right) + \left(x^3 - 3x^2 + 7x + 5\right)\left(\frac{d\left(x^3 - 4\right)}{dx}\right)$$

(Sum & Difference Rules)

$$= \left(\frac{dx^3}{dx} - \frac{d3x^2}{dx} + \frac{d7x}{dx} + \frac{d5}{dx}\right)\left(x^3 - 4\right) + \left(x^3 - 3x^2 + 7x + 5\right)\left(\frac{dx^3}{dx} - \frac{d4}{dx}\right)$$

(Power, Constant Multiple & Constant Rules)

$$= \left(3x^2 - 3\frac{dx^2}{dx} + 7\frac{dx}{dx} + 0\right)\left(x^3 - 4\right) + \left(x^3 - 3x^2 + 7x + 5\right)\left(3x^2 - 0\right)$$

(Power & Identity Rules)

$$= \left(3x^2 - 3(2x) + 7(1)\right)\left(x^3 - 4\right) + \left(x^3 - 3x^2 + 7x + 5\right)\left(3x^2\right)$$

(Minimal arithmetic/algebra cleanup)

$$= \left(3x^2 - 6x + 7\right)\left(x^3 - 4\right) + \left(x^3 - 3x^2 + 7x + 5\right)\left(3x^2\right)$$

(Maximal arithmetic/algebra cleanup)

$$= 6x^5 - 15x^4 + 28x^3 + 3x^2 + 24x - 28$$

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(7) ii. $f(x)=\dfrac{\left(x^3-3x^2+3x+5\right)}{\left(-2x^2+3x-2\right)}$

Solution:

$$f'(x)=\frac{df}{dx}=\frac{d\left(\left(x^3-3x^2+7x+5\right)\left(x^3-4\right)\right)}{dx}$$

(Product Rule)

$$=\left(\frac{d\left(x^3-3x^2+7x+5\right)}{dx}\right)\left(x^3-4\right)+\left(x^3-3x^2+7x+5\right)\left(\frac{d\left(x^3-4\right)}{dx}\right)$$

(Sum & Difference Rules)

$$=\left(\frac{dx^3}{dx}-\frac{d3x^2}{dx}+\frac{d7x}{dx}+\frac{d5}{dx}\right)\left(x^3-4\right)+\left(x^3-3x^2+7x+5\right)\left(\frac{dx^3}{dx}-\frac{d4}{dx}\right)$$

(Power, Constant Multiple & Constant Rules)

$$=\left(3x^2-3\frac{dx^2}{dx}+7\frac{dx}{dx}+0\right)\left(x^3-4\right)+\left(x^3-3x^2+7x+5\right)\left(3x^2-0\right)$$

(Power & Identity Rules)

$$=\left(3x^2-3(2x)+7(1)\right)\left(x^3-4\right)+\left(x^3-3x^2+7x+5\right)\left(3x^2\right)$$

(Minimal arithmetic/algebra cleanup)

$$=\left(3x^2-6x+7\right)\left(x^3-4\right)+\left(x^3-3x^2+7x+5\right)\left(3x^2\right)$$

(Maximal arithmetic/algebra cleanup)

$$=6x^5-15x^4+28x^3+3x^2+24x-28$$

(7) iii. $f(t)=e^{-3t}\sin(2t)$

Solution:

Appendix C - Common Final Exam

$$f'(t) = \frac{df}{dt} = \frac{d\left(e^{-3t}\sin(2t)\right)}{dt}$$

(Product Rule)

$$= \left(\frac{de^{-3t}}{dt}\right)\sin(2t) + e^{-3t}\left(\frac{d\sin(2t)}{dt}\right)$$

(Chain Rule)

$$= \left(\frac{de^{(-3t)}}{d(-3t)} \times \frac{d(-3t)}{dt}\right)\sin(2t) + e^{-3t}\left(\frac{d\sin(2t)}{d(2t)} \times \frac{d(2t)}{dt}\right)$$

(e, Sine & Constant Multiple Rules)

$$= \left(e^{(-3t)} \times -3\frac{dt}{dt}\right)\sin(2t) + e^{-3t}\left(\cos(2t) \times 2\frac{dt}{dt}\right)$$

(Identity Rule)

$$= \left(e^{(-3t)} \times -3(1)\right)\sin(2t) + e^{-3t}\left(\cos(2t) \times 2(1)\right)$$

(Minimal arithmetic/algebra cleanup)

$$= -3e^{(-3t)}\sin(2t) + 2e^{-3t}\cos(2t)$$

(Maximal arithmetic/algebra cleanup)

$$= \left(2\cos(2t) - 3\sin(2t)\right)e^{-3t}$$

(7) iv. $f(x) = \sqrt{\ln(x)} + \cos\left(x^3\right)$

Solution:

$$f'(x) = \frac{df}{dx} = \frac{d\left(\sqrt{\ln(x)} + \cos\left(x^3\right)\right)}{dx} = \frac{d\left((\ln(x))^{1/2} + \cos\left(x^3\right)\right)}{dx}$$

(Sum Rule)

$$= \frac{d\,(\ln(x))^{1/2}}{dx} + \frac{d\cos\left(x^3\right)}{dx}$$

(Chain Rule)

$$= \frac{d\,(\ln(x))^{1/2}}{d\,(\ln(x))} \times \frac{d\,(\ln(x))}{dx} + \frac{d\cos\left(x^3\right)}{d\left(x^3\right)} \times \frac{d\left(x^3\right)}{dx}$$

(Power, ln & Cosine Rules)

$$= \left(\tfrac{1}{2}\right)(\ln(x))^{-1/2} \times \frac{1}{x} + \left(-\sin\left(x^3\right)\right) \times \left(3x^2\right)$$

(Arithmetic/algebra cleanup)

$$= \frac{1}{2x\sqrt{\ln(x)}} - 3x^2\sin\left(x^3\right)$$

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(8) 5. Given $f(x) = x^3 - 3x^2 - 9x$ on the interval [-3,3], use Calculus methods to determine the global (absolute) maximum and minimum values of $f(x)$ on the given interval.

Solution:

The function f is a polynomial, hence continuous everywhere. The given interval, [-3,3] is closed. Thus, to determine global (absolute) extrema we need to: 1) determine all critical numbers within the given interval; 3) compute f at all such critical numbers, as well as the two endpoints, -3 and 3, and see which value is the largest and which is the smallest.

1) $f'(x) = 3x^2 - 3(2x) - 9(1) = 3(x^2 - 2x - 3)$

Thus, $f'(x) = 0 \Leftrightarrow 3(x^2 - 2x - 3) = 0 \Leftrightarrow (x + 1)(x - 3) = 0 \Leftrightarrow x = -1, 3$, both of which are within the given interval.

2) $f(-3) = (-3)^3 - 3(-3)^2 - 9(-3) = -27 - 27 + 27 = -27$

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) = -1 - 3 + 9 = 5$

$f(3) = (3)^3 - 3(3)^2 - 9(3) = 27 - 27 - 27 = -27$

We observe that the global maximum for f on the given interval is 5, occurring at $x = -1$, and the global minimum for f on the given interval is -27, occurring at both $x = -3$ and $x = 3$.

6. Given $x^4 - 2xy^3 + y^5 = 32$:

(5) i. determine $\frac{dy}{dx}$;

(3) ii. determine an equation of the line tangent to the given curve at the point (0,2).

Solutions:

a) Since y is implicitly defined as a function of x we use implicit differentiation.

$$x^4 - 2xy^3 + y^5 = 32 \Rightarrow \frac{d\left(x^4 - 2xy^3 + y^5\right)}{dx} = \frac{d32}{dx}$$

(Sum, Difference & Constant Rules)

$$\Leftrightarrow \frac{dx^4}{dx} - \frac{d2xy^3}{dx} + \frac{dy^5}{dx} = 0$$

(Power, Constant Multiple & Chain Rules)

$$\Leftrightarrow \left(4x^3\right) - 2\frac{dxy^3}{dx} + \frac{dy^5}{dy} \times \frac{dy}{dx} = 0$$

(Product & Power Rules)

$$\Leftrightarrow \left(4x^3\right) - 2\left[\left(\frac{dx}{dx}\right)y^3 + x\left(\frac{dy^3}{dx}\right)\right] + 5y^4\left(\frac{dy}{dx}\right) = 0$$

(Identity & Chain Rules)

$$\Leftrightarrow \left(4x^3\right) - 2\left[(1)y^3 + x\left(\frac{dy^3}{dy} \times \frac{dy}{dx}\right)\right] + 5y^4\left(\frac{dy}{dx}\right) = 0$$

(Power Rule)

$$\Leftrightarrow \left(4x^3\right) - 2\left[y^3 + x\left(3y^2\right)\left(\frac{dy}{dx}\right)\right] + 5y^4\left(\frac{dy}{dx}\right) = 0$$

(Arithmetic/algebra)

$$\Leftrightarrow \left(4x^3\right) - 2y^3 - 6xy^2\left(\frac{dy}{dx}\right) + 5y^4\left(\frac{dy}{dx}\right) = 0 \Leftrightarrow -6xy^2\left(\frac{dy}{dx}\right) + 5y^4\left(\frac{dy}{dx}\right) = 2y^3 - 4x^3$$

(Arithmetic/algebra)

$$\Leftrightarrow \left(-6xy^2 + 5y^4\right)\left(\frac{dy}{dx}\right) = 2y^3 - 4x^3 \Leftrightarrow \left(\frac{dy}{dx}\right) = \frac{\left(2y^3 - 4x^3\right)}{\left(5y^4 - 6xy^2\right)}$$

b) To compute an equation for the line tangent to the given curve at the given point we need the slope of that tangent line. We obtain this by substituting (0,2) for (x,y):

Appendix C - Common Final Exam

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,2)} = \frac{\left(2(2)^3-4(0)^3\right)}{\left(5(2)^4-6(0)(2)^2\right)} = \frac{16}{5(16)} = \frac{1}{5}$$

Now we use the point-slope form of the equation of a straight line:

$$\frac{y-2}{x-0}=\frac{1}{5} \Leftrightarrow \frac{y-2}{x}=\frac{1}{5} \overset{\text{optional}}{\Leftrightarrow} y=\frac{x}{5}+2$$

(7) 7. Determine the derivative of the function $f(x) = 4(x - 1)^{2x}$

Solution:

We note that this function has the form of constant×(variable expression #1)$^{\text{(variable expression \#2)}}$. We know from class that whenever a function has the form (variable expression #1)$^{\text{(variable expression \#2)}}$ the only way to compute the derivative is to use the technique called logarithmic differentiation. We will ignore the constant 4 for now and handle it at the end. That is, suppose $y = (x - 1)^{2x}$. Then $\ln(y) = \ln((x - 1)^{2x}) = 2x \ln(x - 1)$. Now we use implicit differentiation to compute $\frac{dy}{dx}$.

$$\frac{d\ln(y)}{dx}=\frac{d\left(2x\ln(x-1)\right)}{dx}$$

(Chain & Constant Multiple Rules)

$$\Leftrightarrow \frac{d\ln(y)}{dy}\times\frac{dy}{dx}=2\frac{d\left(x\ln(x-1)\right)}{dx}$$

(ln & Product Rules)

$$\Leftrightarrow \frac{1}{y}\times\frac{dy}{dx}=2\left[\left(\frac{dx}{dx}\right)\ln(x-1)+x\left(\frac{d\ln(x-1)}{dx}\right)\right]$$

(Identity & Chain Rules)

$$\Leftrightarrow \frac{1}{y}\times\frac{dy}{dx}=2\left[(1)\ln(x-1)+x\left(\frac{d\ln(x-1)}{d(x-1)}\times\frac{d(x-1)}{dx}\right)\right]$$

(ln & Difference Rules)

$$\Leftrightarrow \frac{dy}{dx}=2\left[\ln(x-1)+x\left(\frac{1}{(x-1)}\times\left[\frac{dx}{dx}-\frac{d1}{dx}\right]\right)\right]y$$

(Identity & Constant Rules)

$$\Leftrightarrow \frac{dy}{dx}=2\left[\ln(x-1)+x\left(\frac{1}{(x-1)}\times[1-0]\right)\right](x-1)^{2x}$$

(Arithmetic/algebra cleanup)

$$\Leftrightarrow \frac{dy}{dx}=2\left[\ln(x-1)+\frac{x}{(x-1)}\right](x-1)^{2x}$$

Thus, using the Constant Multiple Rule we obtain,

$$\frac{df}{dx}=4\left(2\left[\ln(x-1)+\frac{x}{(x-1)}\right](x-1)^{2x}\right)=8\left[\ln(x-1)+\frac{x}{(x-1)}\right](x-1)^{2x}$$

Appendix C - Common Final Exam

(7) 8. Sociologists have determined that crime rates are influenced by temperature. In a midwestern town, the crime rate is modelled by $C=\frac{1}{2}(T+4)^2+80$, where C is the number of crimes per month and T is the average monthly temperature in degrees Celsius. The average temperature for May was 16 degrees, and by the end of May the average temperature was rising at the rate of 5 degrees per month. How fast is the crime rate rising at the end of May?

Solution:

We note that we are given one rate, "by the end of May the average temperature was rising at the rate of 5 degrees per month", or $\left.\frac{dT}{dt}\right|_{t=\text{May}}=5$, the value T(May) = 16, and asked to compute another rate, "How fast is the crime rate rising at the end of May?", or $\left.\frac{dC}{dt}\right|_{t=\text{May}}=\mathbf{?}$. Clearly this is a related rates problem.

Since we are given C as a function of T, we will use the Chain Rule to start:

Since we know the value of both T and $\frac{dT}{dt}$, we substitute in for these and obtain:

$$\frac{dC}{dt}=\frac{dC}{dT}\times\frac{dT}{dt}=\frac{d\left(\frac{1}{2}(T+4)^2+80\right)}{dT}\times\frac{dT}{dt}=\left[\frac{d\left(\frac{1}{2}(T+4)^2\right)}{dT}+\frac{d80}{dT}\right]\frac{dT}{dt}\quad \textbf{(Sum Rule)}$$

$$=\left[\frac{1}{2}\frac{d\left((T+4)^2\right)}{dT}+0\right]\frac{dT}{dt}\quad \textbf{(Chain Rule)}$$

$$=\left[\frac{1}{2}\frac{d\left((T+4)^2\right)}{d(T+4)}\times\frac{d(T+4)}{dT}\right]\frac{dT}{dt}\quad \textbf{(Constant Multiple \& Constant Rules)}$$

$$=\left[\frac{1}{\not{2}}\left(\not{2}(T+4)\right)\times\left(\frac{dT}{dT}+\frac{d4}{dT}\right)\right]\frac{dT}{dt}\quad \textbf{(Power \& Sum Rules)}$$

$$=\left[(T+4)\times(1+0)\right]\frac{dT}{dt}\quad \textbf{(Identity \& Constant Rules)}$$

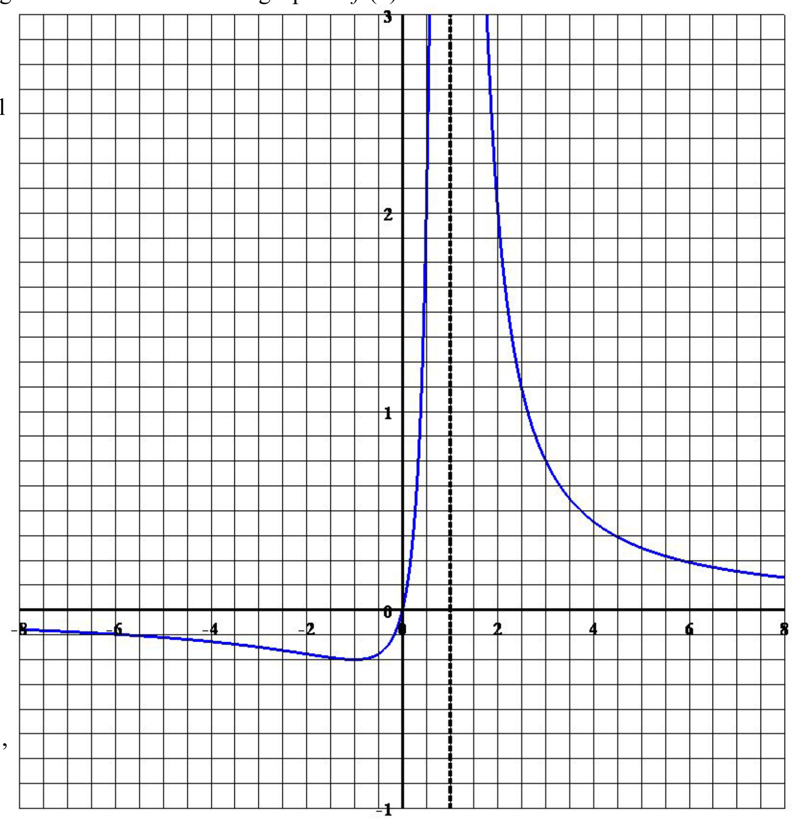
$$=(T+4)\frac{dT}{dt}\quad \textbf{(Arithmetic/algebra cleanup)}$$

So $\left.\frac{dC}{dt}\right|_{t=\text{May}}=\left(T(\text{May})+4\right)\left.\frac{dT}{dt}\right|_{t=\text{May}}=(16+4)5=100$.

That is, at the end of May the crime rate is rising at a rate of 100 crimes per month.

Appendix C - Common Final Exam

(12) 9. Given $f(x)=\frac{x}{(x-1)^2}, f'(x)=-\frac{x+1}{(x-1)^3}, f''(x)=\frac{2x+4}{(x-1)^4}$, use methods learned in this course to determine: any and all horizontal or vertical asymptotes; all local maxima and minima of $f(x)$; all intervals where $f(x)$ is increasing; all intervals where $f(x)$ is decreasing; all points of inflection; all intervals where $f(x)$ is concave up; all intervals where $f(x)$ is concave down. Then use the grid below and sketch a graph of $f(x)$.



Solution:

Step 1: What does $f(x)$ tell us?

$f(0)=\frac{(0)}{((0)-1)^2}=\frac{0}{1}=0$: this tells us that a graph of f will pass through the origin, (0,0).

$f(x)=0 \Leftrightarrow \frac{x}{(x-1)^2}=0 \Leftrightarrow x=0$: this tells us that the only x-intercept for this graph is at $x = 0$, but we already knew of that intercept from calculation above

Is f discontinuous, if yes, where, and what type of discontinuity? Since f is a rational function, and rational functions are continuous where defined, f can only be discontinuous where the denominator is 0. Thus,

$$f(x)=\frac{x}{(x-1)^2} \text{ is discontinuous} \Leftrightarrow x=1$$

To determine what type of discontinuity f has at $x = 1$, we compute the limit of f as x approaches 1.

$$\lim_{x\to1} f(x)=\lim_{x\to1}\left(\frac{x}{(x-1)^2}\right)=\frac{1}{((1)-1)^2}=\text{"}\frac{1}{0}\text{"}$$

Despite the fact that we knew the denominator was 0 at $x = 1$, we substituted in. Because the expression gave us a non-zero number divided by 0, we deduce that nearby the absolute value of f will be very large. That is, when the numerator is close to 1, but the denominator is close to 0, like 0.1, 0.01, 0.001, etc., the ratio is in fact 10, 100, 1000, etc. Since the denominator is squared, it must be non-negative. The numerator is close to 1, hence positive. Thus, all such ratios will be positive, and growing increasingly larger as x approaches 1. This type of discontinuity is referred to as an infinite discontinuity, and on a graph of f, the curve will head upwards on both sides of the vertical line, $x = 1$, which is called a vertical asymptote.

To determine the edge behaviour of f we compute the limit of f as x approaches $\pm\infty$.

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\left(\frac{x}{(x-1)^2}\right)=\lim_{x\to-\infty}\left(\frac{x}{x^2}\right)=\lim_{x\to-\infty}\left(\frac{1}{x}\right)=\text{"}\frac{1}{-\infty}\text{"}=0^-$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{x}{(x-1)^2}\right)=\lim_{x\to\infty}\left(\frac{x}{x^2}\right)=\lim_{x\to\infty}\left(\frac{1}{x}\right)=\text{"}\frac{1}{\infty}\text{"}=0^+$$

The above limits tell us that at both edges a graph of f is asymptotic to the horizontal line, $y = 0$, or the x-axis. Further, we can see that at the left edge the curve will approach the x-axis from below, and on the right edge the curve will approach the x-axis from above.

Step 2: What does $f'(x)$ tell us?

We begin by solving for all critical numbers of f, that is, where f' is either 0 or discontinuous.

$$f'(x)=0 \Leftrightarrow -\frac{x+1}{(x-1)^3}=0 \Leftrightarrow x+1=0 \Leftrightarrow x=-1$$

$$f'(x) \text{ is discontinuous} \Leftrightarrow x-1=0 \Leftrightarrow x=1$$

After entering this information in our table we note that there are three intervals of interest: $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$. We shall use -2, 0 and 2 to represent these three intervals.

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$$f'(-2) = -\frac{(-2)+1}{((-2)-1)^3} = (-)\left(\frac{-}{-}\right) = -$$

$$f'(0) = -\frac{(0)+1}{((0)-1)^3} = (-)\left(\frac{+}{-}\right) = +$$

$$f'(2) = -\frac{(2)+1}{((2)-1)^3} = (-)\left(\frac{+}{+}\right) = -$$

After listing this information in our table we see that f is decreasing on (-∞,-1), has a minimum at x = -1, then increases on (-1,1) and finally decreases on (1,∞). Since f has a minimum value at x = -1, we compute the value of f (-1): $f(-1) = \frac{(-1)}{((-1)-1)^2} = -\frac{1}{4}$

Step 3: What does $f''(x)$ tell us?

We begin by solving for all possible points of inflection of f, that is, where f'' is either 0 or discontinuous.

$$f''(x) = 0 \Leftrightarrow \frac{2x+4}{(x-1)^4} = 0 \Leftrightarrow 2x+4 = 0 \Leftrightarrow x = -2$$

$$f''(x) \text{ is discontinuous} \Leftrightarrow x-1 = 0 \Leftrightarrow x = 1$$

After entering this information in our table we note that there are three intervals of interest: (-∞,-2), (-2,1) and (1,∞). We shall use -3, 0 and 2 to represent these three intervals.

$$f''(-3) = \frac{2(-3)+4}{((-3)-1)^4} = \frac{-}{+} = -; f''(0) = \frac{2(0)+4}{((0)-1)^4} = \frac{+}{+} = +; f''(2) = \frac{2(2)+4}{((2)-1)^4} = \frac{+}{+} = +$$

After listing this information in our table we see that f is concave down on (-∞,-2), has a point of inflection at x = -2, then is concave up on (-2,1) and finally concave down again on (1,∞).

	HA						x-,y-int.			VA			HA
x	-∞		-2		-1		0		1^-	1	1^+		∞
$f(x)$	0^-	↘	↘	↘	-¼ →	↗	0 ↗	↗	∞	U	∞	↘	0^+
$f'(x)$		-	-	-	0	+	+	+		U		-	
$f''(x)$		-	0	+	+	+	+	+		U	+	+	
		⌢		⌣	⌣	⌣	⌣	⌣			⌣	⌣	

Appendix C - Common Final Exam

10. The total cost in dollars of producing x units of a product is given by $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$. When x units are produced the sale price per unit is $p(x) = 1700 - 7x$:

(3) i. determine the revenue function and the profit function;

(5) ii. use methods of Calculus learned in this course to determine the value of x that maximizes the profit and determine the profit for this level of production.

Solution:

a) Total revenue is (price per unit) times (number of units produced). Thus, $TR(x) = p(x)\times x = (1700 - 7x)\times x = 1700x - 7x^2$
Total profit is the difference between total revenue and total cost. Thus,
$TP(x) = (1700x - 7x^2) - (16{,}000 + 500x - 1.6x^2 + 0.004x^3) = -0.004x^3 - 5.4x^2 + 1{,}200x - 16{,}000$

b) The function TP is a cubic or degree three polynomial, hence continuous everywhere. However, the interval of possible values for x is $(0,\infty)$, which is not closed. Thus we must investigate to whether TP has a global (absolute) maximum, and if so, at what value of x the maximum occurs.

$TP'(x) = -0.004(3x^2) - 5.4(2x) + 1{,}200(1) - 0 = -0.012x^2 - 10.8x + 1{,}200$
$= -0.001[12x^2 + 10{,}800x - 1{,}200{,}000] = -0.001\times 12[x^2 + 900x - 100{,}000]$
$= -0.001\times 12(x + 1000)(x - 100)$

We note that $TP'(x) = 0 \Leftrightarrow x = -1000, 100$, but -1000 does not lie in the interval of concern. Since TP is a polynomial, it is continuous everywhere, hence TP has no critical numbers due to TP' being discontinuous. Thus, there are two intervals of concern, (0,100) and (100,∞). We will use 1 to represent the first interval and 200 to represent the second interval.

$TP'(1) = -0.001\times 12((1) + 1000)((1) - 100) = (-)(+)(-) = +$
$TP'(200) = \ = -0.001\times 12((200) + 1000)((200) - 100) = (-1)(+)(+) = -$

			Max		
x	0		100		∞
$TP(x)$		↗ ↗ ↗	→	↘ ↘ ↘	
$TP'(x)$		+ + + + +	0	- - - - -	

From the table we can see that TP has a global maximum value when $x = 100$.
That maximum value would be:
$TP(100) = -0.004(100)^3 - 5.4(100)^2 + 1{,}200(100) - 16{,}000$
$= -4000 - 54{,}000 + 120{,}000 - 16{,}000 = 46{,}000$
That is, when the level of production is 100 units, the maximum profit of $46,000 is achieved.

Appendix D - Coding & Scoring Schema

Coding & Scoring Schema for Term Test 1 Common Questions:

Exponential growth or continuously compounded interest: solve for value of *t* (independent variable)	Code	Score
1. Use correct formula and substitute correct values Use correct formula and substitute incorrect values Use wrong exponential formula and substitute given values correctly Use any other formula	4 3 2 0	$\frac{\text{Code}}{4}$
2. If 1. = 0 then stop grading this problem, otherwise If the formula is rewritten in one or more steps to an equation with *t* no longer an exponent, and one could solve for *t* (this should involve ln) and no algebraic error algebraic error(s) If the formula is **incorrectly** rewritten in one or more steps to an equation with *t* no longer an exponent, and one could solve for *t*	 4 2 0	$\frac{\text{Code}}{4}$
3. Correct final result for *t* (any precision) using correct formula and correct substitution Incorrect final result for *t* or any result based on either incorrect original formula or correct original formula with incorrect substitution	2 0	$\frac{\text{Code}}{2}$

Problem on compound interest: solve for rate of interest (*r* or *i*)	Code	Score
1. Use correct formula and substitute correct values Use correct formula and substitute incorrect values Use wrong exponential formula and substitute given values correctly Use non-exponential formula	4 3 2 0	$\frac{\text{Code}}{4}$
2. If 1. = 0 then stop grading this problem, otherwise If the formula is correctly rewritten in one or more steps to a linear equation solving for *r* If the exponent is removed with no logarithm remaining on *r*, but there are algebra/arithmetic error(s) If the exponent is removed incorrectly due to function errors, or *r* is not solved for	 4 2 0	$\frac{\text{Code}}{4}$
3. Correct final result for *r* (any precision) using correct formula and correct substitution Incorrect final result for *r* or any result based on either incorrect original formula or correct original formula with incorrect substitution	2 0	$\frac{\text{Code}}{2}$

Appendix D - Coding & Scoring Schema

Coding & Scoring Schema for Term Test 2 Common Questions:

In all questions: 1 = correct, 0 = incorrect, 99 = nothing shown in this category
Concerning algebra/arithmetic errors: if something, *e.g.*, a denominator, is missing, but the expression before and after is correct, then we do not count this omission as an error.

	Limit Question: Continuous point	Score
1.	Correct limit syntax: (no limit without function, no function without limit, no limit without $x\rightarrow$ beneath, no use of $\Rightarrow$ where there should be =) (99 possible)	0
2.	All algebra and arithmetic is correct (99 possible)	1
3.	There is a verbal indication that the correct answer for the limit is 0. (99 possible)	2
4.	Correct answer (correct numerical value is 0, not 0/# and no additional statements indicating erroneous reasoning)	1
5.	Graph consistent with answer (note that empty grid is considered a graph) (99 possible)	1
6.	Graph correct (any curve passing through the appropriate solid *x*-intercept) (note that empty grid is considered a graph) (99 possible)	2
	Limit Question: Infinite discontinuity	**Score**
1.	Correct limit syntax: (no limit without function, no function without limit, limit without x-> beneath, no use of -> where it should be =) (99 possible)	0
2.	All algebra and arithmetic is correct (99 possible)	1
3.	Presence (1) or absence (0) of half-limit computation (either symbolically or numerically, even one side only)	1
4.	Correct answer: DNE (does not exist), U (undefined) or appropriate half-limits are $-\infty$ and $+\infty$ (and no other answers that are wrong nor additional statements indicating erroneous reasoning)	2
5.	Graph consistent with answer (note that empty grid is considered a graph) (99 possible)	1
6.	Graph correct (vertical asymptote with sides in correct direction) (note that empty grid is considered a graph) (99 possible)	2
	Limit Question: Removable discontinuity	**Score**
1.	Correct limit syntax: (no limit without function, no function without limit, limit without x-> beneath, no use of -> where it should be =) (99 possible)	0
2.	All algebra and arithmetic is correct (99 possible)	1
3.	Correct answer (correct numerical value, and no other answers that are wrong nor additional statements indicating erroneous reasoning)	2
4.	Graph consistent with answer (note that empty grid is considered a graph) (99 possible)	1
5.	Graph correct (curve through a hollow point for correct answer) (note that empty grid is considered a graph) (99 possible)	2

Appendix D - Coding & Scoring Schema

Newton's Quotient (a)		Score
1.	Correct limit syntax: (no limit without function, no function without limit, limit without x-> beneath, no use of -> where it should be =) (99 possible)	0
2.	Correct representation of $f\,(x + h)$ (99 possible)	1
3.	All algebra and arithmetic is correct (99 possible)	1
4.	Differentiation rules were not used (other than as a check to completed Newton's Method)?	0
5.	Did they finish the technique correctly to the end? (99 possible)	3

"Used" in the context below means more than just computed, but actually used in computation of a tangent line equation.

Newton's Quotient (b)		Score
1.	If the value used for $f\,(a)$ is that of $f\,'(a)$ then code 0, otherwise code 1. (99 possible)	0
2.	Is the value used for $f\,(a)$ correct? (99 possible)	2
3.	Is the value used for $f\,'(a)$ is the slope of the linear function $f\,'(x)$ code 0, otherwise code 1. (99 possible)	0
4.	Is the value used for $f\,'(a)$ correct? (99 possible)	2
5.	Did they use point-slope form of equation of a line?	0
6.	Is their final equation correct? (99 possible)	3
Chain Rule (a)		Score
1.	Did they correctly replace p with $p(t)$? (99 possible)	2
Chain Rule (b)		Score
1.	Did they replace (anywhere) t with its correct numerical value? (99 possible)	1
2.	Did they use the correct function in computing $Q(a)$? (99 possible)	1
3.	Did they compute $Q(a)$ correctly (their answer agrees with teacher answer)? (99 possible)	2
Chain Rule (c)		Score
1.	Did they attempt to compute the derivative? If no, then 2. - 9. below are coded 99.	1
2.	Did they use a version of the function with a negative power when attempting to compute the derivative? (99 possible)	0
3.	If yes to 2. above, was the version correct? (99 possible)	0
4.	Did they attempt to use the chain rule where it was applicable? (99 possible)	1
5.	If the answer to 4. above was yes, did they perform the chain rule correctly? (99 possible)	2
6.	The quotient rule was not used. (99 possible only if they did not attempt to compute the derivative, or in part (a) there was no quotient)	0
7.	Did they get the correct expression for the derivative (agree with teacher answer)? (99 possible)	2
8.	Did they replace the value into the above expression? (99 possible)	1
9.	Was all their numerical computation of the value of $Q'(a)$ correct, given their expression for $Q'(a)$? (99 possible)	2

Appendix D - Coding & Scoring Schema

Chain Rule (d)		Score
1.	Did they correctly interpret the sign of whatever quantity they computed for $Q'(a)$? (99 possible)	3

Coding & Scoring Schema for Term Test 3 Common Questions:

In all questions: 1 = correct, 0 = incorrect, **99 = 0 on primary question (indicated by P)**.
On questions where 99 is possible you can only code 99 if primary question has been coded as 0.
Concerning algebra/arithmetic errors: if something, *e.g.*, a denominator, is missing, but the expression before and after is correct, then we do not count this omission as an error.

Question 1a: Product Rule		Score
1.P	Product rule correctly implemented (ignore errors in power rule, trig. diff., notation or algebra/arithmetic)?	1
2.	Derivative of trigonometric function correctly given? (99 possible)	1
3.	Power rule correctly implemented (ignore algebra errors and incorrectly handling the negative exponent)? (99 possible)	1
4.	Student correctly decreased negative exponent in power rule (*e.g.*, $\frac{dx^{-9}}{dx} = -9x^{-10}$ **not** $\frac{dx^{-9}}{dx} = -9x^{-8}$)? (99 possible)	1
5.	Neither algebra nor notational errors present? (99 possible)	0
6.P	Is student answer correct?	3
Question 1b: Quotient Rule		**Score**
1.P	Quotient rule correctly implemented (ignore errors in exponential rule, trig. diff., notation or algebra/arithmetic)?	1
2.	Derivative of trigonometric function correctly given? (99 possible)	1
3.	Derivative of exponential function correctly given? (99 possible)	1
4.	Neither algebra nor notational errors present? (99 possible)	0
5.P	Is student answer correct?	3

Appendix D - Coding & Scoring Schema

Question 1c: Logarithmic Differentiation		Score
Solution Method 1 1.P	 Logarithmic differentiation was attempted?	0
2.	Was log applied to both sides? (99 possible)	0
3.	Was the right hand side converted to a product of exponent times logarithm of base? (99 possible)	0
4.	Was chain rule correctly implemented on the LHS? (99 possible)	2
5.	Was product rule correctly implemented on the RHS? (99 possible)	1
6.	Derivative of the log function on the RHS correct? (99 possible)	2
Solution Method 2 7.P	 Was change of base attempted?	0
8.	Change of base implemented correctly? (99 possible)	2
9.	Was chain rule implemented correctly? (99 possible)	2
10.	Was product rule inside chain rule implemented correctly? (99 possible)	1
Solution by either Method 11.P	 Is student answer correct?	3

Question 2: Implicit Differentiation		Score
1.P	Did the student understand that y must be seen as a function of x (in at least one instance y was differentiated with respect to x)?	1
2.P	Was product rule implemented correctly on the product, $x^m y^n$, regardless of use/non-use of chain rule?	1
3.P	Was the chain rule correctly implemented for powers of y which stand alone (disregarding multiplied constants)?	1
4.P	Was the chain rule correctly implemented for the power of y which is multiplied times a power of x?	1
5.P	Was there an attempt to isolate y'?	0
6.	Was y' isolated correctly (within their work)? (99 possible)	1
7.P	Was the value of y' correct?	3
8.P	Was there an attempt to compute the slope at (1,-1) from the formula for y'?	0
9.	Was the slope computed correctly (within their work)? (99 possible)	1
10.P	Was the tangent line equation correct?	2

Appendix D - Coding & Scoring Schema

Question 3: Demand Function		Score
1.P	Was chain rule used in differentiation of D as a function of t (*i.e.*, do you see a product of $\frac{dD}{dp}$ and $\frac{dp}{dt}$, whether symbolically or numerically)?	1
2.P	Was the derivative $\frac{dD}{dp}$ computed correctly (up to algebraic error)?	3
3.P	Was there any algebraic or notational error?	1
4.P	Was the numerical answer correct?	3
5.P	Was the sign of their numerical answer interpreted correctly in terms of inc./dec. of D (if there was no numerical answer, code this as 0, no matter what choice of inc./dec.)?	2

Appendix D - Coding & Scoring Schema

Question 4: Graphing		Score
1.P	If the student wrote absolutely nothing, code = 0 and stop coding.	0
2.P	Did they determine the critical point correctly?	1
3.	Did they correctly identify the critical point as a min./max. (based on any evidence of first or second derivative test, but not on graph, and even if they never explicitly use the word min. or max.) (99 possible)	1
4.	Did they correctly identify intervals of increasing/decreasing (based on any evidence, *e.g.*, table, other than graph)? (99 possible)	1
5.P	Did they determine the inflection point correctly?	1
6.	Did they correctly identify intervals of concave up/down (based on any evidence, *e.g.*, table, other than graph)? (99 possible)	1
7.P	Did they attempt to calculate $\lim_{x\to-\infty} f(x)$ (numerical arguments are acceptable)?	0
8.	Did the student correctly relate this left edge limit to the absence/presence of a horizontal asymptote at the left edge (must be an explicit statement of no HA, but can be through listing in table, but not from graph)? (99 possible)	1
9.P	Did they attempt to calculate $\lim_{x\to+\infty} f(x)$ (numerical arguments are acceptable)?	0
10.	Did the student correctly relate this right edge limit to the absence/presence of a horizontal asymptote (must be an explicit statement of HA, but can be through listing in table, but not from graph)? (99 possible)	1
11.P	Is any part of the curve depicted in a graph? (if coded as 0, 12 - 17 left blank)	0
12.	Left edge behaviour portrayed on graph consistently with previous work? (99 if no previous work)	1
13.	Right edge behaviour portrayed on graph consistently with previous work? (99 if no previous work)	1
14.	Inflection point on graph, in the neighbourhood (±1) of their computed point and looks like an inflection point? (99 if no previous work)	1
15.	Maximum/minimum point on graph, in the neighbourhood (±1) of their computed point and looks like one? (99 if no previous work)	1
16.	No features appear on graph that did not appear in work? (such as extra extrema, inflection points or asymptotes).	1
17.	Is graph correct (up to y-axis scaling issues)?	3

Appendix D - Coding & Scoring Schema

Coding & Scoring Schema for Common Final Exam:

In all questions: 1 = correct, 0 = incorrect, **99 = 0 on primary questions (indicated by P)**.
On questions where 99 is possible you can only code 99 if the primary question has been coded as 0.
Concerning algebra/arithmetic errors: if something, *e.g.*, a denominator, is missing temporarily, but the expression before and after the omission is correct, then we do not count this omission as an error.
If a student has crossed out/erased work, it is to be counted as attempted. If this work is correct, and nothing else has been written to replace it, then it counts as the student's work. If something else has been written to replace it, then that replacement work counts as the student's final work.

Question 1a: Limit	Score
1.P Was factorization of the numerator and denominator attempted?	1
2. Was the factorization correct? (99 possible)	1
3.P Was a numerical table around the limiting point attempted?	or 1
4.P Was the final numerical answer correct?	1
Question 1b: Infinite limit	Score
1.P Was a numerical table around the limiting point attempted?	1
2.P Did the student go correctly beyond DNE or equivalent (*i.e.*, indicates vertical asymptote or infinite discontinuity or correctly shows both half-limits)?	1
3.P Was the student answer correct (DNE with no stupid error is considered correct)?	2
Question 2: Continuity	Score
1.P Was the (dis-)continuity correctly identified algebraically at the left point of interest?	1
2.P Was the (dis-)continuity correctly identified algebraically at the right point of interest?	1
3.P Was no extra point of discontinuity included?	1
4.P Was any graph attempted (anything beyond a single point or vertical line(s))?	0
5. Was the (dis-)continuity of the left point of interest correctly represented on the graph? (99 possible)	2
6. Was the (dis-)continuity of the right point of interest correctly represented on the graph? (99 possible)	2

Appendix D - Coding & Scoring Schema

Question 3: Newton's Quotient	Score
1.P Was any development of the RHS of the Newton's Quotient formula, as given, attempted?	0
2. Was limit syntax used correct? (99 possible)	1
3. Was the $x+h$ correctly substituted into $f(x)$ for x to represent $f(x+h)$? (99 possible)	1
4. No algebra/arithmetic errors were present? (99 possible)	0
5.P Was the technique correct all the way to the final result for the derivative?	2
6.P Did the student use $f(x)$, and substitute the given x-value, to compute the y-value of the point of tangency?	1
7.P Did the student use $f'(x)$, and substitute the given x-value, to compute the slope of the tangent line (if no evidence of use of $f'(a)$ as slope, then code as 0)?	1
8.P Was the final equation for the tangent line correct?	2
Question 4a: Product Rule	Score
1.P Was the product rule attempted?	0
2. Was the product rule correctly implemented? (99 possible)	2
3.P No power rule errors were present?	1
4. Were there no algebra errors present prior to any expansion of bracketed terms? (99 possible)	1
5.P Was the final result correct?	2
Question 4b: Quotient Rule	Score
1.P Was the quotient rule attempted?	0
2. Was the quotient rule correctly implemented? (99 possible)	2
3.P No power rule or algebra errors were present?	1
4. Were there no algebra errors present prior to any expansion of bracketed terms? (99 possible)	1
5.P Was the final result correct?	2
Question 4c: Product Rule with Exp Trig functions	Score
1.P Was the product rule attempted?	0
2. Was the product rule correctly implemented? (99 possible)	1
3.P Was the exponential rule correctly implemented (ignore chain rule)?	2
4.P Was the trigonometric rule correctly implemented (ignore chain rule)?	2
5.P Was the final result correct?	2

Appendix D - Coding & Scoring Schema

Question 4d: Chain Rule	Score
1.P Was the ln term rewritten as a composition of ln(x) and a power function (power function on the outside)?	1
2. Was the chain rule correctly implemented in the first term? (99 possible)	2
3.P Was chain rule correctly implemented in the second term?	2
4.P Was the final result correct?	2
Question 5: Global min/max	**Score**
1.P Was the derivative correct?	1
2.P Was there an attempt to determine the critical points?	1
3. Was $f'(x)=0$ solved correctly? (99 possible)	2
4.P Were values of f computed at critical points?	1
5.P Were values of f computed at both interval end points?	1
6.P Was the correct global maximum (y-value) determined?	1
7.P Was the correct global minimum (y-value) determined?	1
Question 6: Implicit Differentiation	**Score**
1.P Did $\frac{dy}{dx}$ or y' occur **only** in appropriate places? (doesn't occur at all, or appears in inappropriate places, then code as 0).	1
2.P Was product rule implemented correctly on the product, $x^m y^n$, regardless of use/non-use of chain rule?	1
3.P Was the chain rule correctly implemented for powers of y which stand alone (disregarding multiplied constants)?	1
4.P Was the chain rule correctly implemented for the power of y which is multiplied times a power of x?	1
5.P Was there an attempt to isolate y'?	0
6.P Was the value of y' correct?	1
7.P Was the slope computed correctly (within their work)?	1
8.P Was the tangent line equation correct (within their work)?	2

Appendix D - Coding & Scoring Schema

Question 7: Logarithmic Differentiation **Note: 3. - 11. are to be scored neglecting how the multiplied constant on the RHS is handled. For example, the answer is correct even if the constant is handled incorrectly.**	Score
Solution Method 1 1.P Was ln applied to both sides?	1
2. Was multiplied constant on RHS handled correctly? (99 possible)	0
3. Was the right hand side converted to a product of exponent times logarithm of base? (99 possible)	1
4. Was the chain rule correctly implemented on the LHS? (99 possible)	1
5. Was the product rule correctly implemented on the RHS? (99 possible)	1
6. Was the derivative of the log function on the RHS correct? (99 possible)	1
Solution Method 2 7.P Was change of base attempted?	1
8. Was multiplied constant on RHS handled correctly? (99 possible)	0
9. Was change of base implemented correctly? (99 possible)	1
10. Was the chain rule implemented correctly? (99 possible)	2
11. Was the product rule inside the chain rule implemented correctly? (99 possible)	1
Solution by either Method 12.P Was the student answer correct?	2

Question 8: Related Rates	Score
1.P Was the chain rule used in differentiation of C as a function of t (*i.e.*, do you see a product of $\frac{dC}{dT}$ **and** $\frac{dT}{dt}$, whether symbolically or numerically)?	3
2.P Was the derivative $\frac{dC}{dT}$ computed correctly (up to algebraic error)?	3
3.P Was the numerical answer correct?	2

Appendix D - Coding & Scoring Schema

Question 9: Graphing	Score
1.P If the student wrote absolutely nothing, code = 0 and stop coding.	0
2. Did they not include the discontinuity of f' in the critical points? (99 possible)	0
3. Did they determine the critical point(s) correctly? (99 possible)	1
4. Did they correctly identify the critical point as a minimum/maximum (based on any evidence of first or second derivative test, but not on the graph, and even if they never explicitly use the word minimum or maximum) (99 possible)	1
5. Did they check for increasing/decreasing on three correct intervals? (99 possible)	0
6. Did they correctly identify intervals of increasing/decreasing based on any evidence, *e.g.*, table, other than the graph? (99 possible)	1
7.P Did they determine the inflection point correctly?	1
8. Did they check concavity on three correct intervals? (99 possible)	0
9. Did they correctly identify intervals of concave up/down based on any evidence, *e.g.*, table, other than the graph? (99 possible)	1
10.P Did they attempt to calculate $\lim_{x\to-\infty} f(x)$ (numerical arguments are acceptable)?	0
11. Did the student correctly relate this left edge limit to the absence/presence of a horizontal asymptote at the left edge (must be an explicit statement of no HA, but can be through listing in table, but not from the graph)? (99 possible)	1
12.P Did they attempt to calculate $\lim_{x\to+\infty} f(x)$ (numerical arguments are acceptable)?	0
13. Did the student correctly relate this right edge limit to the absence/presence of a horizontal asymptote (must be an explicit statement of HA, but can be through listing in table, but not from graph)? (99 possible)	1
14.P Did the student correctly identify the VA?	1
15.P Was any part of the curve depicted in a graph? (if coded as 0, 12 - 17 left blank)	0
16. Was the left edge behaviour portrayed on the graph consistently with previous work? (99 if no previous work)	1
17. Was the right edge behaviour portrayed on the graph consistently with previous work? (99 if no previous work)	1
18. Was the VA portrayed on the graph consistently with previous work? (99 if no previous work)	1
19. Is there an identifiable inflection point on graph, in the neighbourhood of their computed point? (99 if no previous work)	1
20. Is there an identifiable maximum/minimum point on graph, in the neighbourhood of their computed point? (99 if no previous work)	1
21. No additional features appeared on the graph that did not appear in work? (such as extra extrema, inflection points or asymptotes).	1
22. Was the graph correct (up to y-axis scaling issues)?	1

Appendix D - Coding & Scoring Schema

Question 10: Optimization	Score
1.P Was the revenue function correctly determined from the demand function?	1
2.P Was the profit function correctly determined as P=R-C?	1
3.P Was the derivative of P (or of R and C) attempted?	0
4. Was P ′ (or R′ and C′) correct? (99 possible)	1
5. Was P′ = 0 (or R′ = C′) solved correctly? (99 possible)	2
6. Was the *x*-value of the maximum correctly identified (within their work)? (99 possible)	1
7.P Was maximum profit correctly determined?	1

Appendix E - Common Assignment Problems

Paper Assignment #1 with Solutions

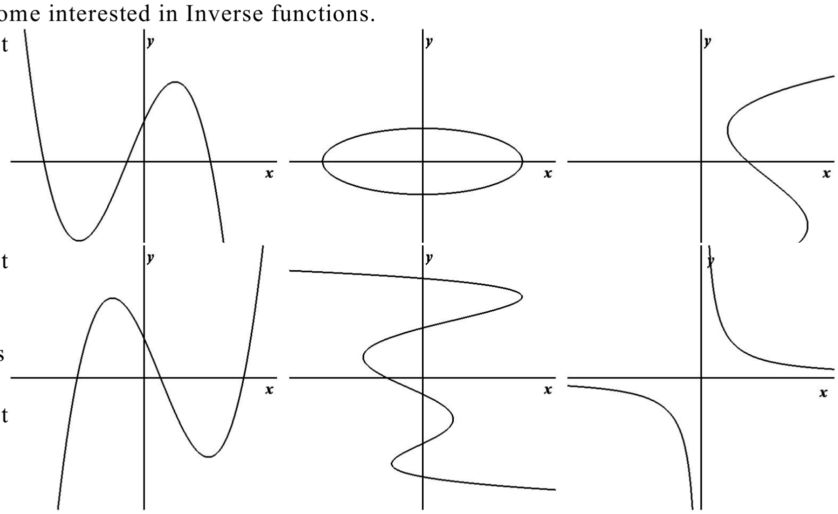
1. For each of the graphs below indicate (Yes or No) in the appropriate space whether the corresponding curve defines y as a function of x.

Solutions:

One definition of a function is: a rule or correspondence between two sets (called Domain and Range) whereby each element of the Domain set is assigned to a **unique** element of the Range set.

I bolded the word **unique** to emphasize it since it is the important distinction between a function and what is called a "relation" in mathematics. A relation may assign multiple values of the Range set to correspond to a single element of the Domain, where a function assigns a **unique** value. When we draw graphs of functions, by convention the Domain of the function is part or all of the horizontal axis while the Range is part of all of the vertical axis. Thus, to distinguish between graphs of relations and graphs of functions, for each value of x (element of the Domain) we imagine the vertical line with that value and we determine if the line intersects the graph. If a vertical line does not intersect the graph, then that x is not in the Domain. If **any** vertical line intersects the graph at more than one point, then the graph is of a relation, not of a function. This imagined drawing or visualization of all possible vertical lines is not very imaginatively called the Vertical Line Test for a Function. Later in the course we will come across another "imaginatively" named test called the Horizontal Line Test when we become interested in Inverse functions.

a. Yes - no vertical line would intersect more than once.
b. No - we can visualize many vertical lines that would intersect twice.
c. No - we can visualize many vertical lines that would intersect three times.
d. Yes - no vertical line would intersect more than once.
e. No - we can visualize many vertical lines that would intersect as many as five times.
f. Yes - no vertical line would intersect more than once.



d. e. f.

2. Provincial sales tax y is directly proportional to retail price x. An item that sells for \$172 has a sales tax of \$14.42 dollars. Determine a mathematical model that gives the amount of sales tax y in terms of the retail price x.

Solutions:

If y is directly proportional to x, then $y = ax$ (Eq. #1), for some constant a. Since we were given that when $x = 172$, that $y = 14.42$, if we substitute these into (Eq. #1) we obtain: $14.42 = a \times 172$ (Eq. #2).

Now, dividing both sides of Eq. #2 by 172 we obtain: $(14.42/172) = a \Leftrightarrow a = 0.083837$, so

Your answer is: $\boldsymbol{y = 0.083837\, x}$

What is the sales tax on a \$330 purchase?

To compute the answer to this we just replace x in the above formula by \$330 and calculate y: $y = 0.083837 \times 330 = 27.66628$

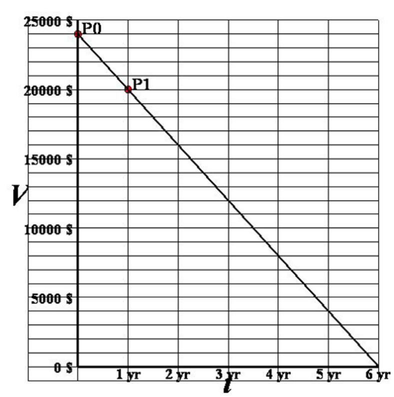
Your answer is: **\$27.67**

3. The value of a police vehicle is expected to decrease over time following a linear model. The graph below represents the value V of a vehicle at time t, where V is measured in \$ and t is measured in years from the time of purchase.

What would point P_0 on the line represent (in words in the real world)?

What would point P_1 on the line represent (in words in the real world)?

The slope-intercept equation which conveys the relationship between the value of a police vehicle V to time t is:



Solutions:

In mathematics we refer to the point P_0 as the y-intercept of the line. However, in the real world P_0 is just the initial value of the car, immediately upon purchasing it.

In the real world P_1 is the value of the car one year after purchasing it.

Appendix E - Common Assignment Problems

The points P_0 (0,24000) and P_1 (1,20000) lie on the line. Hence the slope of the line, using the usual formula for slope, $m = \dfrac{y_1 - y_0}{x_1 - x_0}$,

would be: $m = \dfrac{20000-24000}{1-0} = \dfrac{-4000}{1} = -4000$. This would be the coefficient of *t* in the answer. We already have the constant term, which is the same as the *y*-intercept value, 24000. Hence our answer is:

V = **-4000** t **+ 24000**

4. The equation of the line that passes through the points (-5, -1) and (8, 1) can be written in the form $y = mx + b$

Solutions:

We use the standard formula for slope, $m = \dfrac{y_1 - y_0}{x_1 - x_0}$, and obtain: $m = \dfrac{1-(-1)}{8-(-5)} = \dfrac{2}{13}$.

Substituting this value for m into the standard $y = mx + b$ equation, as well as the values of the point (8, 1), we obtain:

$y = mx + b \Rightarrow 1 = \left(\dfrac{2}{13}\right)\times 8 + b \Leftrightarrow 1 - \left(\dfrac{16}{13}\right) = b \Leftrightarrow b = \dfrac{-3}{13}$ where m = **(2/13)** and b = **-(3/13)**

5. An equation for the line with slope -3 that goes through the point (7, -7) can be written in point slope form as:

Solutions:

We are given that slope is -3, so m = -3, and that the line goes through the point (7, -7), so x_0 = 7 and y_0 = -7.

$\dfrac{y - y_0}{x - x_0} = m$, where m = **-3**, and x_0 = **7** and y_0 = **-7**

If we want the *y*-intercept as well, we algebraically manipulate the function defining equation and obtain *b*:

$\dfrac{y-(-7)}{x-7} = -3 \Leftrightarrow \dfrac{y+7}{x-7} = -3 \Leftrightarrow y + 7 = -3(x-7) \Leftrightarrow y = -3x + 21 - 7 \Leftrightarrow y = -3x + 14$ and also in slope-*y*-intercept form as:

$y = mx + b$, where m = **-3** and b = **14**.

6. An equation for the line with *x*-intercept 3 and *y*-intercept 6 can be written in the form

Solutions:

The *x*-intercept being three tells us that the point (3, 0) lies on the line. The *y*-intercept being 6 tells us that the point (0, 6) lies on the line. Using the standard formula for slope, given two points, we obtain: $m = \dfrac{y_1 - y_0}{x_1 - x_0} \Rightarrow m = \dfrac{6-0}{0-3} = \dfrac{6}{-3} = -2$.

Since *b* represents the *y*-intercept, and we were given that is 6, we now have:

$y = mx + b$, where m = **-2** and b = **6**

7. The population of a certain town in increasing rapidly. In 1982 the population numbered 11 thousand residents and in 1996 the number of residents was 36 thousand. Assume that the population is described by a linear function of time, $P(t) = kt + b$, where *t* is time since 1980.

Solutions:

In this example *k* represents the slope of the line and *b* represents population at year 0, or the *y*-intercept.

We were given that (2, 11) and (16, 36) lie on the line. Using the standard formula for slope, $m = \dfrac{y_1 - y_0}{x_1 - x_0} \Rightarrow m = \dfrac{36-11}{16-2} = \dfrac{25}{14}$.

If $t = 0$ corresponds to the year 1980 then k = (25/14) in thousands of people per year and substituting in the information concerning *k* as well as one of the given points, say (2, 11), into the equation $P(t) = kt + b$, we obtain:

$$P(t) = kt + b \Rightarrow 36 = \left(\frac{25}{14}\right)16 + b \Rightarrow b = \frac{104}{14} = 7.4286$$

b = **7.4286** in thousands of people.

Note that a negative population just does not make sense. This negative answer just indicates that the assumption that population is described by a linear function is not reasonable.

How large will the population of this town be in the year 2010 if the growth continues to follow the same linear model,

Appendix E - Common Assignment Problems

To compute this just substitute in $t = 30$: $P(30) = \left(\frac{25}{14}\right)(30) + \frac{104}{14} = \frac{854}{14} = \mathbf{61}$, *i.e.*, $P(30) =$ **61** in thousands of people.

8. In the 90's annual attendance at professional baseball games was decreasing. Figures for 1992 to 1997 are shown in the table below. Determine the average rate of change (to three decimal places) in annual attendance between:

Year	1992	1993	1994	1995	1996	1997
Attendance (millions)	25.6	24.8	24.5	24	23.1	22.5

Solutions:
Average rate of change, like slope of a line connecting two points, is just the ratio of (change in dependent variable) divided by (change in independent variable), or (Δy / Δx). With this in mind
we compute the three average rates of change requested:

$$\frac{24.5 - 25.6}{1994 - 1992} = \frac{-1.1}{2} = -0.55 \qquad \frac{22.5 - 24.5}{1997 - 1994} = \frac{-2.0}{3} = -0.667 \qquad \frac{22.5 - 25.6}{1997 - 1992} = \frac{-3.1}{5} = -0.62$$

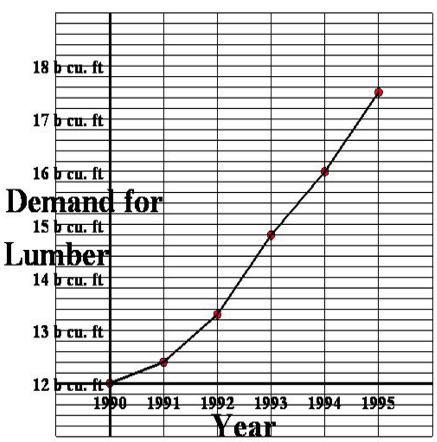
1992 to 1994: **-0.550** 1994 to 1997: **-0.667** 1992 to 1997: **-0.62**

9. The graph and data table below indicate the data points which were gathered regarding annual demand for timber from commercial forest land in the 1990's. Demand is measured in billions of cubic feet of lumber. Demand is seen to be increasing, but not a constant rate. Determine (to two decimal place accuracy) the average rate of change in demand between:

Solutions:
Average rate of change, like slope of a line connecting two points, is just the ratio of (change in dependent variable) divided by (change in independent variable), or (Δy / Δx). With this in mind
we compute the three average rates of change requested:

$$\frac{14.8 - 12}{1993 - 1990} = \frac{2.8}{3} = 0.933 \qquad \frac{17.5 - 14.8}{1995 - 1993} = \frac{2.7}{2} = 1.35 \qquad \frac{17.5 - 12}{1995 - 1990} = \frac{5.5}{5} = 1.1$$

1990 and 1993: **0.933** 1993 and 1995: **1.35** 1990 and 1995: **1.1**



Year	Demand (in billions ft^3)	Increase in Demand
1990	12	
1991	12.4	0.4
1992	13.3	0.9
1993	14.8	1.5
1994	16	1.2
1995	17.5	1.5

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10. You are driving from Montreal to Toronto but your speedometer is broken so you cannot easily tell your speed (non-directional average rate of change). This can be expensive since you have already noticed several Ontario Provincial Police cars with radar guns enforcing the 100 km/hr speed limit, and the fines for speeding in Ontario, posted prominently on billboards that you have passed, are high. Usually this would be no problem, you just "go with the flow", adjusting your speed to that of other cars on the road, but eerily there are no other cars on the road. Your friend, sitting next to you in the front seat, volunteers to estimate your speed by doing rate of change calculations. Using her watch and kilometrage markers posted along the highway, she gathers the following data:

km marker	122.8	122.9	123.0	123.1	123.2	123.3
time (seconds)	0	3.5	8	12.3	16.5	20.6

What was your average speed (four decimal places) from:

Solutions:

Speed is just the name for average rate of change when the dependent variable is distance travelled and the independent variable is time. Average rate of change, like slope of a line connecting two points, is just the ratio of (change in dependent variable) divided by (change in independent variable), or (Δy / Δx). With this in mind we compute the five average rates of change requested:

$$\frac{123.0-122.8}{8-0}=\frac{0.2}{8}=0.025 \quad \frac{123.0-122.9}{8-3.5}=\frac{0.1}{4.5}=0.0222 \quad \frac{123.1-123.0}{12.3-8}=\frac{0.1}{4.3}=0.0233 \quad \frac{123.2-123.0}{16.5-8}=\frac{0.2}{8.5}=0.0235$$

$$\frac{123.3-123.0}{20.6-8}=\frac{0.3}{12.6}=0.0238$$

km 122.8 to 123.0: **0.0250** km/s km 122.9 to 123.0: **0.0222** km/s
km 123.0 to 123.1: **0.0233** km/s km 123.0 to 123.2: **0.0235** km/s
km 123.0 to 123.3: **0.0238** km/s

The best estimate would most likely be obtained by averaging over the interval 122.9 to 131.1. This is because this is the shortest interval (so speed has less time to vary) that includes numbers on both sides of 123.0. We can do this calculation directly (as done above), or by averaging the two average rates of change already obtained for 122.9 to 123.0 and 123.0 to 123.1. Both methods will arrive at the same answer: $\frac{123.1-122.9}{12.3-3.5}=\frac{0.2}{8.8}=0.0227$

All of the above are estimates of your exact speed at km marker 123.0. What is your best estimate of that speed (four decimal places): **0.0227** km/s

Changing units is a common problem wherever mathematics is used. Here is a simple technique that always works.

We want to change from seconds to hours. Our first step is to determine two quantities, measured in these two units, that are equal. In this case, 60×60 seconds = 1 hr. Since these quantities are equal, if we form a ratio where one quantity is the numerator and the other is the denominator, the ratio will have numerator = denominator, so the ratio is really just a fancy way of writing 1. The two possible ratio's here are: $\frac{3600\,\text{s}}{1\,\text{hr}}$ or $\frac{1\,\text{hr}}{3600\,\text{s}}$. Now it is well known that multiplying by 1 does not change whatever number you multiply, so we may multiply by either of these forms of 1. We choose the first form for this problem since it allows us to cancel the s unit in the denominator of the original answer and have hr instead. Thus,

$$0.0227\,\frac{\text{km}}{\text{s}}\times 3600\,\frac{\text{s}}{\text{hr}}=81.8182\,\frac{\text{km}}{\text{hr}}$$

Now change your estimate to common units: **81.8182** km/hr

11. The position of a cat running from a dog down a dark alley is given by the values of the table.

t (seconds)	0	1	2	3	4	5
s (feet)	0	8	27	73	93	118

Solution:

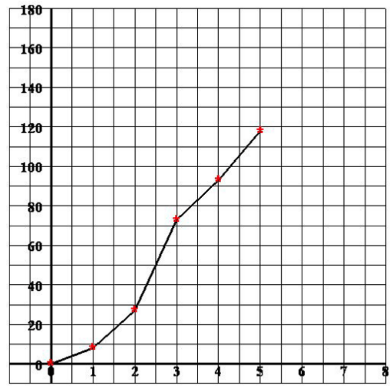
Average velocity or speed is just the name for average rate of change when the dependent variable is distance travelled and the independent variable is time. Average rate of change, like slope of a line connecting two points, is just the ratio of (change in dependent variable) divided by (change in independent variable), or (Δy / Δx). With this in mind we compute the three average rates of change requested:

Appendix E - Common Assignment Problems

$\frac{118-27}{5-2}=\frac{91}{3}=30.3333$ $\frac{93-27}{4-2}=\frac{66}{2}=33$ $\frac{73-27}{3-2}=\frac{46}{1}=46$

a. Determine the average velocity (speed) of the cat (ft/sec) for the time period beginning when $t = 2$ and lasting:
   i) 3 s: **30.3333** ii) 2 s: **33** iii) 1 s: **46**

b. Draw the graph of the function for yourself.



12. The following table shows the "living wage" jobs in Chicoutimi per 1000 working age adults over a 5 year period.

Year	1997	1998	1999	2000	2001
Jobs	625	680	720	755	785

Solutions:
Average rate of change, like slope of a line connecting two points, is just the ratio of (change in dependent variable) divided by (change in independent variable), or (Δy / Δx). With this in mind we compute the two average rates of change requested:

$\frac{720-625}{1999-1997}=\frac{95}{2}=42.5$ $\frac{785-720}{2001-1999}=\frac{65}{2}=32.5$

a. What is the average rate of change in the number of living wage jobs from 1997 to 1999?
   42.5 Jobs/Year

b. What is the average rate of change in the number of living wage jobs from 1999 to 2001?
   32.5 Jobs/Year

Based on these two answers, should the mayor from the last two years be reelected? (These numbers are made up. Please do not actually hold the mayor accountable.)

It appears that the mayor for the last two years has not been as successful as his predecessor in creating jobs. Perhaps the previous mayor had a better strategy for job creation and is a better candidate.

13. Below is a table of values for some unknown function $f(t)$.
   Calculate the average rate of change of $f(t)$ on the interval $0.5 \le t \le 2$

Solution:
Average rate of change, like slope of a line connecting two points, is just the ratio of (change in dependent variable) divided by (change in independent variable), or (Δy / Δx). With this in mind we compute the average rate of change requested:

$\frac{11.8-38.05}{2-0.5}=\frac{-26.25}{1.5}=-17.5$

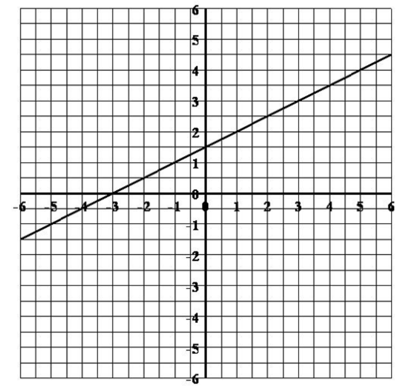
Average rate of change = **-17.5**

t	0.5	1	1.5	2	2.5	3
$f(t)$	38.05	36.8	28.05	11.8	-12	-43.2

Appendix E - Common Assignment Problems

14. As you know, for a linear function $f(x) = mx + b$, m is the slope and b is the y-intercept of the line. Changing m while holding b fixed will generate many lines of different slopes, all passing through the same y-intercept. Changing b while holding m fixed will generate many lines having the same slope, hence parallel, but each with different y-intercept.
    Draw a graph of the line that passes through the points (-3,0) and (3,3). Without computing anything but just observing the graph answer the following questions:

Solutions:



The rate of change of a linear function is the same as the slope. (Note that only linear functions have the property that no matter what x-interval you use to calculate average rate of change (slope), you get the same answer.) The easiest way to see this is to realize that the formula for average rate of change is exactly the same as for the slope of the line connecting two points. When observing the graph of a linear function, we can read off the slope as "rise over run". In this example it is easy to observe that the function is "running" twice as fast as it is "rising", hence the average rate of change is just ½. For example, starting at (-3,0) we run two units to the right and then up one unit to arrive at (-1,1), which is clearly on the line.

a. What is the rate of change of this function: **½**

Again, by observation of the graph we note that the line crosses the y-axis at (0, 3/2), so the y-intercept is **3/2**.

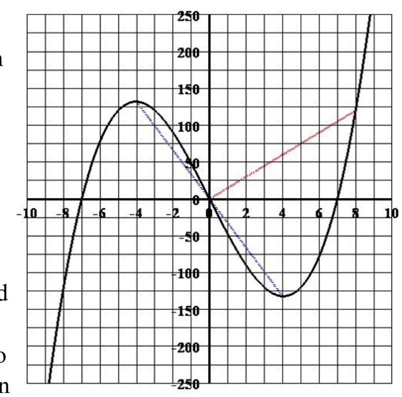
b. What is the y-intercept: **3/2**

An x-intercept of a function $f(x)$ is any x-value such that $f(x) = 0$, *i.e.*, any x-value where a graph of f intersects the x-axis. Other than horizontal lines (and this line is not horizontal), all lines have only a single x-intercept. In this case, we were actually given the point of intersection of the line and the x-axis, namely (-3, 0). Thus, the y-intercept of this line is just -3.

c. What is the x-intercept: **-3**

15. The graph below-right is of some unknown function.

Solutions:



Suppose we call the function $f(x)$. Then, to compute the average rate of change of this function on the x-axis interval (0,8) is essentially the same as computing the slope connecting two points on the graph, namely $(0, f(0))$ and $(8, f(8))$. We observe from the graph that $f(0) = 0$ and $f(8) = 125$ (halfway between the horizontal lines for $y = 100$ and $y = 150$). Thus, the average rate of change requested, or the slope, is just: $\frac{f(8)-f(0)}{8-0} = \frac{125-0}{8} = \frac{125}{8} = 15.625$

a. What is the average rate of change on the x-axis interval (0,8)? **15.625**

To compute the average rate of change on the x-axis interval (-4,4) we need to determine $f(-4)$ and $f(4)$. Looking at the given graph we note that unlike $f(0)$ and $f(8)$ which were vertices on the graph, hence easily read, we will have to estimate these two values. Of course, if I connect the two points on the graph with a straight line, and the straight line passes through two vertices, then I can use those two points and probably get a more accurate estimate (assuming I am good at drawing straight lines). In fact, when we look at the line there are three such points on the line, (-3, 100), (0, 0) and (3, -100). Using the latter two we obtain the slope:

$$\frac{-100-0}{3-0} = \frac{-100}{3} = -33.\overline{3}$$

b. What is the average rate of change on the x-axis interval (-4,4)? $\mathbf{-33.\overline{3}}$

Appendix E - Common Assignment Problems

Paper Assignment #2 with Solutions

Instructions: Some problems involve "Yes/No" type answers, or multiple choice answers, and for these your answer must be indicated in the appropriate blank (_____). In problems that involve calculation, show your work and then circle your answer and copy it to the appropriate blank.

1. A long term investment of $250,000 has been made by a young widow. The interest is 12% per year (i = 0.12), and interest is compounded quarterly. If the interest rate remains at 12% per year, to the nearest whole dollar what will the value of the investment be after:

Solutions:

The formula for compound interest investments is: $V(t) = P\left(1+\frac{i}{n}\right)^{nt}$, where V is the value of the investment at time t, P is the principal, or amount initially invested, i is the annual projected rate of interest (APR) and n is the number of times per year that interest is compounded.

In this example the first sentence (A long term investment of $250,00 ...) tells us that P = 250,000, The first phrase and the bracketed expression in the second sentence (The interest is 12% per year (i = 0.12)) tells us that i = 0.12, and the second phrase (interest is compounded quarterly) tells us the the formula above is appropriate for this problem and that n = 4. The last phrase in the last sentence (what will the value of the investment be ...) tells us that we are being asked to solve for V, and the values listed below are all values of t. Thus, we can see that if we substitute the known values into the formula we can compute V as follows:

$$V(1) = 250000\left(1+\frac{0.12}{4}\right)^{4(1)} = 250000(1+0.03)^4 = 250000(1.03)^4 = 281377.2025$$

$$V(3) = 250000\left(1+\frac{0.12}{4}\right)^{4(3)} = 250000(1+0.03)^{12} = 250000(1.03)^{12} = 356440.221711545$$

$$V(5) = 250000\left(1+\frac{0.12}{4}\right)^{4(5)} = 250000(1.03)^{20} = 451527.808667354$$

$$V(10) = 250000\left(1+\frac{0.12}{4}\right)^{4(10)} = 250000(1.03)^{40} = 815509.44799977$$

$$V(12) = 250000\left(1+\frac{0.12}{4}\right)^{4(12)} = 250000(1.03)^{48} = 1033062.96981504$$

1 year: **$281,377** 3 years: **$356,440** 5 years: **$451,528** 10 years: **$815,509** 12 years: **$1,033,063**

2. If $6000 is invested at an interest rate of of 8% per year, compounded semiannually, determine the value of the investment after the given number of years.

Solutions:

This problem is essentially the same as the first, but this time P = 6000, i = .08, and n = 2.

The calculation is as follows:

$$V(5) = 6000\left(1+\frac{0.08}{2}\right)^{2(5)} = 6000(1+0.04)^{10} = 6000(1.04)^{10} = 8881.4657$$

$$V(10) = 6000\left(1+\frac{0.08}{2}\right)^{2(10)} = 6000(1+0.04)^{20} = 6000(1.04)^{20} = 13146.7389$$

$$V(15) = 6000\left(1+\frac{0.08}{2}\right)^{2(15)} = 6000(1+0.04)^{30} = 6000(1.04)^{30} = 19460.3851$$

5 years: **$8,881** 10 years: **$13,147** 15 years: **$19,460**

Appendix E - Common Assignment Problems

3. The population of the world in 1987 was 5 billion and the growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, determine the projected world population in 2007.

Solution:

The phrase "follows an exponential growth model" immediately tells you that the function this problem is describing is $P(t) = Ce^{kt}$ (formula #1), where P is world population, C and k are unknown constants (vaguely like the m and b of $y = mx + b$), and t is time. In such problems we are usually given two pieces of information, equivalent to two points on the graph of this function, and expected to use those pieces of information to determine the two unknown constants. Once the constants are known, given any value of either P or t, we can compute the corresponding value of t or P respectively.

To make calculation simpler in this example (and noting that the answer is requested in billions below) we measure P in billions. Similarly, to make the t time values simpler we let 1987 be $t = 0$. Thus the phrase "T1.he population of the world in 1987 was 5 billion" translates into mathematical language as $P(0) = 5$, rather than the uglier $P(1987) = 5{,}000{,}000{,}000$. But substituting $t = 0$ into formula #1 yields: $P(0) = Ce^{k(0)} = Ce^0 = C\cdot 1 = C$. Thus, we already know $C = 5$, so our model function is now $P(t) = 5e^{kt}$ (formula #2). The phrase "the growth rate was estimated at 2 percent per year" suggests that $P(1) = 1.02\times 5 = 5.1$. Substituting $t = 1$ into formula #2 from above, we obtain: $P(1) = 5e^{k(1)} = 5e^k$. Thus, we can see that:

$$5.1 = 5e^k \Leftrightarrow e^k = \frac{5.1}{5} = 1.02 \Leftrightarrow \ln\left(e^k\right) = \ln(1.02) \Leftrightarrow k = \ln(1.02)$$

Comment: Although we solved for k above, strictly speaking this is not necessary. Stopping at $e^k = 1.02$ is simpler and will get us to the same answer, so this is what is shown below.

If we observe that formula #2 can be rewritten as $P(t) = 5(e^k)^t$, we can now substitute 1.02 for e^k and obtain: $P(t) = 5(1.02)^t$ (formula #3).

The phrase "determine the projected world population in 2007" tells us that the value asked for in this problem is $P(20)$ (if $t = 0$ when the year is 1987, then $t = 20$ when the year is 2007 since 2007 - 1987 = 20). Substituting $t = 20$ into formula #3 we obtain: $P(20) = 5(1.02)^{20} \doteq 7.4297$. This then is our answer.

My answer is **7.4297** billion.

4. In class and in the textbook we have seen sample graphs of exponential functions, $f(x) = b^x$., $0 < b$. What is the effect on the graph when you change the value of the base, b?

Without computing anything, just by thinking about the effect of having different b values, indicate whether each of the following statements is true or false.

Solutions:

If the base of an exponential function is bigger than 1, then the exponential function is said to be a "growth" function, and a graph of the function shows that it is increasing (from left to right) over the entire x-axis. The reason for this is seen clearly with a small data table. Suppose our function is $f(x) = b^x$, where as an example of $b > 1$ we choose $b = 2$. Then

x	-3		-2		-1		0		1		2		3
$f(x)$	$2^{-3} = 1/8$		$2^{-2} = ¼$		$2^{-1} = ½$		$2^0 = 1$		$2^1 = 2$		$2^2 = 4$		$2^3 = 8$

We can see clearly why as x values move left to right across the table, so do the $f(x)$ values as well. Thus, the statement a. is True

a. **True** All exponential functions with base bigger than 1 have positive rates of change on every interval.

If we set $b = 1$, then $f(x) = 1^x = 1$, no matter the value of x. For example, $1^{-2} = 1/1^2 = 1/1 = 1$ or $1^0 = 1$ or $1^3 = 1$. Thus, the statement b. is True.

b. **True** The exponential function with base 1 is constant and so of no interest.

If you have looked at the graph of any "growth" exponential function, that is, those with base bigger than 1, you will have observed that all of them are concave up everywhere as well as increasing everywhere (statement a. above). Checking this is more difficult if we are not looking at graphs, just the formula. In part this is because the standard definition of concavity depends upon knowledge of material that only comes later in this course. However, there is an alternative definition of concavity that can be used at this point. A function $f(x)$ is said to be concave up on an x-interval (a,b) if for every x_1 and x_2, where $a < x_1 < x_2 < b$, the secant line passing through $(x_1,f(x_1))$ and $(x_2,f(x_2))$ lies above $f(x)$ on the x-interval (x_1,x_2). Using this definition to prove the statement involves difficult Algebra (shown below) so that most of us are satisfied by looking at pictures. Thus, we mark the statement c. as True as well.

Note: The following is overkill, hence optional, and only for those who are interested in what a proof would look like.

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$f(x)=b^x,\ b>1,\ x_1<x<x_2$ **so** $f(x_1)=b^{x_1}<f(x)=b^x<f(x_2)=b^{x_2}$

(the last just says in symbols that an exponential with base >1 is increasing)

Secant line connecting $(x_1,f(x_1))$ **to** $(x_2,f(x_2))$:

$$\frac{y-f(x_1)}{x-x_1}=\frac{f(x_2)-f(x_1)}{x_2-x_1}\Leftrightarrow y=\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)(x-x_1)+f(x_1)$$

$$\Leftrightarrow y=\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)x-\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)x_1+f(x_1)$$

Thus, the statement that at x **the secant line lies above the curve translates into symbols as:**

$$y>f(x)\Leftrightarrow\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)x-\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)x_1+f(x_1)>b^x$$

$$\Leftrightarrow\left(\frac{b^{x_2}-b^{x_1}}{x_2-x_1}\right)x-\left(\frac{b^{x_2}-b^{x_1}}{x_2-x_1}\right)x_1+b^{x_1}>b^x\Leftrightarrow\left(\frac{b^{x_2}-b^{x_1}}{x_2-x_1}\right)(x-x_1)>b^x-b^{x_1}$$

$$\Leftrightarrow\left(\frac{b^{x_2}-b^{x_1}}{x_2-x_1}\right)>\frac{b^x-b^{x_1}}{(x-x_1)}\qquad \textbf{(I\#1)}$$

But $\left(\frac{b^{x_2}-b^{x_1}}{x_2-x_1}\right)$ **is just the slope of the line connecting** $(x_1,f(x_1))$ **to** $(x_2,f(x_2))$ **and**

$\frac{b^x-b^{x_1}}{(x-x_1)}$ **is just the slope of the line connecting** $(x_1,f(x_1))$ **to** $(x,f(x))$.

But $x<x_2$, **and** $f(x)<f(x_2)$, ***i.e.*****, the point** $(x,f(x))$ **is below and to the left of the point** $(x_2,f(x_2))$ **so the the slope of the line connecting** $(x_1,f(x_1))$ **to** $(x_2,f(x_2))$ **must be less than the slope of the line connecting** $(x_1,f(x_1))$ **to** $(x,f(x))$,

which is just what our final inequality (I#1) says.

c. **True** All exponential functions with base bigger than 1 are concave up everywhere.

If you have looked at the graph of any "decay" exponential function, that is, those with base smaller than 1, you will have observed that all of them are concave up everywhere as well as decreasing everywhere. Checking this is more difficult if we are not looking at graphs, just the formula, but it is similar to the proof given above, so those who are interested can do it for themselves, those who are not interested can just ignore this and use the evidence of their eyes. Thus, our answer for statement d. is False.

d. **False** All exponential functions with base between 0 and 1 are concave down everywhere.

5. John just inherited \$20,000.00 unexpectedly. He would like to use the money to buy a new car, but the car that he wants to buy costs \$30,000.00. John decides to invest the money, compounded monthly, at interest rate *i*, and wait two years. What interest rate *i* (expressed as a percentage and up to two decimal places) would John need so that at the end of the two years he has \$30,000?

Solution:

The formula for compound interest investments is: [illegible], where V is the value of the investment at time t, P is the principal, or amount initially invested, i is the annual projected rate of interest (APR) and n is the number of times per year that interest is compounded.

In this example the phrase "John decides to invest the money, compounded monthly" tells us that $n = 12$. The initial sentence tells us that $P = 20000$ and the final phrase "at the end of the two years he has \$30,000" tells us that $t = 2$ and $V(2) = 30000$. The phrase "What interest rate *i*" tells us that we must solve for the value of *i*.

We begin by substituting the four known values into the compound interest formula and then solve for *i*:

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$$30000 = 20000\left(1+\frac{i}{12}\right)^{12(2)} \Leftrightarrow 30000 = 20000\left(1+\frac{i}{12}\right)^{24} \quad \text{(divide both sides by 20000)}$$

$$\frac{3\cancel{0000}}{2\cancel{0000}} = \frac{\cancel{20000}}{\cancel{20000}}\left(1+\frac{i}{12}\right)^{24} \Leftrightarrow 1.5 = \left(1+\frac{i}{12}\right)^{24} \quad \text{(take ( )}^{1/24}\text{ on both sides)}$$

$$(1.5)^{1/24} = \left(\left(1+\frac{i}{12}\right)^{\cancel{24}}\right)^{\cancel{1/24}} \Leftrightarrow (1.5)^{1/24} = 1+\frac{i}{12} \quad \text{(subtract 1 from both sides)}$$

$$(1.5)^{1/24} - 1 = \cancel{1} + \frac{i}{12} \cancel{-1} \Leftrightarrow (1.5)^{1/24} - 1 = \frac{i}{12} \quad \text{(multiply both sides by 12)}$$

$$\left((1.5)^{1/24} - 1\right)\times 12 = \left(\frac{i}{\cancel{12}}\right)\times \cancel{12} \Leftrightarrow i = \left((1.5)^{1/24} - 1\right)\times 12 \doteq 0.204454759267043$$

Thus, we can see that the interest rate required is approximately 20.45%.

i = **20.45%**

6. In class and in the textbook we have seen graphs of the function ln(x). On your calculator you will find a button (same button used for e^x) for ln(x).

Solutions:

The average rate of change is the same as the slope of a secant line connecting two points on a graph of the function. Because in this case the function is ln(x), all points on the graph have the form (x,ln(x)). When the question talks about the x-interval (0.5,1), it just means that the two points are (0.5,ln(0.5)) and (1,ln(1)). We should know by heart that ln(1) = 0 so this second point is (1,0). Using a calculator or spreadsheet or some computer program we obtain ln(0.5) ≐ -0.6931, so our first point is (0.5,-0.6931). Using the standard formula for slope of a line connecting two given points we obtain:

$$m \doteq \frac{0-(-0.6931)}{1-0.5} = \frac{0.6931}{\left(\frac{1}{2}\right)} = 0.6931\times\frac{2}{1} = 1.3862$$

a. What is the average rate of change on the interval (0.5,1)? **1.3862**

This is the same problem, but the points are (1,0) and (2,ln(2))≐(2,0.6931)

Note: It is no accident that ln(-½) = -0.6931 and ln(2) = 0.6931. Do you know why?

Using the standard formula for slope of a line connecting two given points we obtain:

$$m \doteq \frac{0.6931-0}{2-1} = \frac{0.6931}{1} = 0.6931$$

b. What is the average rate of change on the interval (1,2)? **0.6931**

Answer the following two questions either Yes or No:

If we have observed a graph of ln(x), even just once, then it is clear that statement c. is true.

c. Is the function ln(x) increasing everywhere it is defined? **Yes**

Again, if we have observed a graph of ln(x), even just once, then it is clear that statement d. is true.

d. Is the function ln(x) concave down? **Yes**

Think of what your answer to this last question means about the growth rate of this function ln(x), is it accelerating or slowing down?

Looking at a graph of ln(x) shows us that the rate of growth of the function ln(x) is in fact slowing down. That is, the slopes of secant lines get smaller as the two points used move to the right. Our two calculations already show us the way with the slope on (0.5,1) being approximately 1.3862, while the slope on (1,2) is approximately 0.6931.

7. Express each of the given exponential equations in logarithmic form:

Solutions:

The key "rules" in problems that involve moving between exponential equations and logarithmic equations are the identities that spring from the fact that exponentials and logarithms are inverse functions of each other. Namely, $\log_a(a^{(\ )}) = (\ )$, which converts exponentials into logarithms and $a^{\log_a(\ )} = (\ )$, which converts logarithms to exponentials.

In this example, we use the first of these to convert the given exponential equation to a logarithmic one:

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$$4^5 = 1024 \Rightarrow \log_4\left(4^5\right) = \log_4\left(1024\right) \Leftrightarrow 5 = \log_4\left(1024\right)$$

Looking at the pattern for the answers that we are expected to provide, $\log_4 A = B$, we see that $A = 1024$ and $B = 5$ so

a. Given $4^5 = 1024$, rewrite this exponential equation in the form $\log_4 A = B$, where
A = **1024** and B = **5**

We use the same technique, but with logarithms base 10 instead of base 4 and obtain"

$$10^{-4} = 0.0001 \Rightarrow \log_{10}\left(10^{-4}\right) = \log_{10}\left(0.0001\right) \Leftrightarrow -4 = \log_{10}\left(0.0001\right)$$

Looking at the pattern for the answers that we are expected to provide, $\log_{10} C = D$, we see that $C = 0.0001$ and $D = -4$ so

b. Given $10^{-4} = 0.0001$, rewrite this exponential equation in the form $\log_{10} C = D$,
where C = **0.0001** and D = **-4**

8. Express each of the given logarithmic equations in exponential form:

Solutions:

As stated above in problem 7, we need to use $a^{\log_a(\ )} = (\ \ )$, or rather in this case $e^{\ln(\ )} = (\ \ )$ which converts logarithms base e to exponentials base e.

$$\ln(5) = x \Rightarrow e^{\ln(5)} = e^x \Leftrightarrow 5 = e^x$$

Looking at the pattern for the answers that we are expected to provide, $e^A = B$, we note that $A = x$ and $B = 5$. Thus,

a. Given $\ln(5) = x$, rewrite this exponential equation in the form $e^A = B$, where
A = $\boldsymbol{x}$ and B = **5**

We use the same technique here, but starting with a different equation, and obtain:

$$\ln(x) = 5 \Rightarrow e^{\ln(x)} = e^5 \Leftrightarrow x = e^5$$

Looking at the pattern for the answers that we are expected to provide, $e^C = D$, we note that $C = 5$ and $D = x$. Thus,

b. Given $\ln(x) = 5$, rewrite this exponential equation in the form $e^C = D$, where
C = **5** and D = $\boldsymbol{x}$

9. If $\log_b(2) = x$ and $\log_b(3) = y$, evaluate the following in terms of x and y:

Solutions:

These type of problems involve remembering rules of logarithms:

$\log_b(xy) = \log_b(x) + \log_b(y)$
$\log_b(x/y) = \log_b(x) - \log_b(y)$
$\log_b(x^y) = y\log_b(x)$

Now we note that $54 = 2\times27 = 2\times3^3$ so $\log_b(54) = \log_b(2\times3^3) = \log_b(2) + \log_b(3^3) = \log_b(2) + 3\log_b(3)$. Now, using the given information that $\log_b(2) = x$ and $\log_b(3) = y$ we obtain $\log_b(54) = x + 3y$

a. $\log_b(54)$ = $\boldsymbol{x + 3y}$

This time we note that $108 = 2\times54 = 2^2\times3^3$ so:
$\log_b(108) = \log_b(2^2\times3^3) = \log_b(2^2) + \log_b(3^3) = 2\log_b(2)+3\log_b(2)$
Now, using the given information that $\log_b(2) = x$ and $\log_b(3) = y$ we obtain $\log_b(108) = 2x + 3y$

b. $\log_b(108)$ = $\boldsymbol{2x + 3y}$

This time we note that $16 = 2^4$ and $27 = 3^3$, so:

$$\log_b\left(\frac{16}{27}\right) = \log_b\left(16\right) - \log_b\left(27\right) = \log_b\left(2^4\right) - \log_b\left(3^3\right) = 4\log_b\left(2\right) - 3\log_b\left(3\right) = 4x - 3y$$

c. $\log_b\left(\frac{16}{27}\right)$ = $\boldsymbol{4x - 3y}$

This time we note that $81 = 3^4$ and $16 = 2^4$, so $\dfrac{\log_b(81)}{\log_b(16)} = \dfrac{\log_b\left(3^4\right)}{\log_b\left(2^4\right)} = \dfrac{\cancel{4}\log_b(3)}{\cancel{4}\log_b(2)} = \dfrac{y}{x}$

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d. $\dfrac{\log_b(81)}{\log_b(16)} = \dfrac{y}{x}$

Note that a common error is to write $\dfrac{\log_b(81)}{\log_b(16)} = \log_b(81) - \log_b(16)$, as if there was a rule that said $\dfrac{\log_b(x)}{\log_b(y)} = \log_b(x) - \log_b(y)$. There is no such rule. The rule that looks vaguely like this is $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$. Be careful to notice the difference!

10. Determine the value of x:
Solutions:
As in problem 7., the key "rules" in problems that involve moving between exponential equations and logarithmic equations are the identities that spring from the fact that exponentials and logarithms are inverse functions of each other. Namely, $\log_a(a^{(\ )}) = (\ \ )$, which converts exponentials into logarithms and $a^{\log_a(\ )} = (\ \ )$, which converts logarithms to exponentials.
In a. we are given a logarithmic equation, so we use the second of these identities. That is,

$$\log_x(343) = 3 \Rightarrow x^{\log_x(343)} = x^3 \Leftrightarrow 343 = x^3 \Rightarrow (343)^{1/3} = \left(x^3\right)^{1/3} \Leftrightarrow x = (343)^{1/3} = 7$$

Note, it is nice if you notice in advance that 343 = 7^3, but not absolutely necessary. Noting it would allow you to reduce the original LHS: $\log_x(343) = \log_x(7^3) = 3\log_x(7)$ so that the equation to solve is just $3\log_x(7) = 3 \Leftrightarrow \log_x(7) = 1$, but $3\log_x(x) = 1$, so $x = 7$. This solution is slightly simpler, but requires that first observation.

a. $\log_x(343) = 3 \Rightarrow x =$ **7**

This is a similar problem, but we immediately notice that $25 = 5^2$, and so use the following:

$\log_x(25) = 2 \Leftrightarrow \log_x(5^2) = 2 \Leftrightarrow 2\log_x(5) = 2 \Leftrightarrow \log_x(5) = 1 \Leftrightarrow x = 5$ (as reasoned in the note above)

b. $\log_x(25) = 2 \Rightarrow x =$ **5**

11. Determine the solution of the exponential equation, $2^{\left(-\frac{x}{9}\right)} = 15$, in terms of logarithms,
Solution:
In this example, we use $\log_2(2^{(\ )}) = (\ \ )$ to convert the given exponential equation to a logarithmic one:

$$2^{\left(-\frac{x}{9}\right)} = 15 \Rightarrow \log_2\left(2^{\left(-\frac{x}{9}\right)}\right) = \log_2(15) \Leftrightarrow \left(-\frac{x}{9}\right)\log_2(2) = \log_2(15)$$

$$\Leftrightarrow \left(-\frac{x}{9}\right) \times 1 = \log_2(15) \Leftrightarrow x = -9\log_2(15) \doteq -35.1620$$

or correct to four decimal places: $x =$ **-35.1620**

12. Solve each of the following for x:
Solution:
Since the bases here are different on both sides of the equations we use base e, *i.e.*, ln( ), for our own convenience. We also use properties of logarithms, namely: $\ln(x^y) = y\ln(x)$. Thus, we obtain:

$9^x = 2^{(x+1)} \Rightarrow \ln(9^x) = \ln(2^{(x+1)}) \Leftrightarrow x\ln(9) = (x+1)\ln(2) \Leftrightarrow x\ln(9) = x\ln(2) + \ln(2)$
$\Leftrightarrow x(\ln(9) - \ln(2)) = \ln(2) \Leftrightarrow x = \ln(2)/(\ln(9/2)) \doteq 0.4608$

a. $9^x = 2^{(x+1)} \Rightarrow x =$ **0.4608**

This problem is similar, but we must use one additional property, $\ln(xy) = \ln(x) + \ln(y)$. Thus, we obtain:

$3(2^x) = 11^x \Rightarrow \ln(3(2^x)) = \ln(11^x) \Leftrightarrow \ln(3) + \ln(2^x) = x\ln(11) \Leftrightarrow \ln(3) + x\ln(2) = x\ln(11)$
$\Leftrightarrow x\ln(2) - x\ln(11) = -\ln(3) \Leftrightarrow x(\ln(2) - \ln(11)) = -\ln(3) \Leftrightarrow x = -\ln(3)/(\ln(2) - \ln(11)) \doteq 0.6444$

b. $3(2^x) = 11^x \Rightarrow x =$ **0.6444**

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13. Solve each of the following for x:

For these problems we use the following notions: $e^{\ln(\ )} = (\ \ )$; $\ln(xy) = \ln(x) + \ln(y)$; $\ln(x/y) = \ln(x) - \ln(y)$; and, $\ln(1) = 0$. Using some of these we obtain:

$$\ln(x-10)+\ln(x+10)=0 \Leftrightarrow \ln\left((x-10)(x+10)\right)=\ln(1) \Leftrightarrow e^{\ln((x-10)(x+10))}=e^{\ln(1)}$$
$$\Leftrightarrow (x-10)(x+10)=1 \Leftrightarrow x^2-100=1 \Leftrightarrow x^2=101 \Leftrightarrow x=\pm\sqrt{101} \doteq \pm 10.0498756211209$$

a. $\ln(x - 10) + \ln(x + 10) = 0 \Rightarrow x =$ **±10.0499**

Similarly, we obtain:

$$\ln(x-1)-\ln(2x-12)=0 \Leftrightarrow \ln\left(\frac{x-1}{2x-12}\right)=\ln(1) \Leftrightarrow e^{\ln\left(\frac{x-1}{2x-12}\right)}=e^{\ln(1)}$$
$$\Leftrightarrow \left(\frac{x-1}{2x-12}\right)=1 \Leftrightarrow x-1=2x-12 \Leftrightarrow -1+12=2x-x \Leftrightarrow x=11$$

b. $\ln(x - 1) - \ln(2x - 12) = 0 \Rightarrow x =$ **11**

14. In H. G. Wells famous novel "When the Sleeper Wakes", a man sleeps for centuries while his wisely invested trust fund essentially ends up owning most of the world. Suppose that you set up a trust fund that started with $1,000, invested at a rate of interest, i, that lets it double in exactly 6 years.

Solutions:

The question below mentions that the interest is continuously compounded. This means that the formula for compound interest investments that we use is a simple one: $V(t) = Pe^{it}$, where V is the value of the investment at time t, P is the principal, or amount initially invested and i is the annual rate of interest.

The phrase "started with $1,000" tells us that $P = 1000$. The phrase "lets it double in exactly 6 years" tells us that $V(6) = 2000$. Putting this information into our formula we obtain:

$$2000 = 1000e^{i6} \text{ (divide both sides by 1000) } \Leftrightarrow \frac{2\cancel{000}}{1\cancel{000}} = \frac{\cancel{1000}}{\cancel{1000}}e^{i6}$$
$$\Leftrightarrow 2 = e^{i6} \text{ (take ln( ) on both sides)} \Leftrightarrow \ln(2)=\ln\left(e^{i6}\right) \Leftrightarrow \ln(2)=6i \text{ (divide both sides by 6)}$$
$$\Leftrightarrow \frac{\ln(2)}{6}=\frac{\cancel{6}i}{\cancel{6}} \Leftrightarrow i=\frac{\ln(2)}{6} \doteq 0.115524530093324$$

From the above we can see that our answer to question a. is $i = 11.55\%$

a. What interest rate i (expressed as a percentage and up to two decimal places) would guarantee that, assuming continuous compounding?

$i =$ **11.55%**

Since we know that the money doubles every 6 years, the question really is how many 6 year periods there are in 200 years or 2000 years? Clearly we obtain these simply by dividing 200 or 2000 by 6. For each such 6 year period the original 1000 doubles, *i.e.*, is multiplied by a factor of 2. Thus, after 200 years the value of the trust is $1000 \times 2^{200/6} \doteq 10{,}822{,}639{,}409{,}680.90$ or 1.0822×10^{13}. Note that this is about 11 trillion dollars. To give some perspective, the Gross Domestic Product (GDP) of the United States, as listed in the online version of the CIA Factbook, is estimated at $\$12.36 \times 10^{12}$ for 2005.

b. Without using the interest rate i computed above, determine how much money (give a number in the form *.**** × 10^{*}) would accumulate for your descendants (or you if you are cryogenically frozen and then reawakened, for you) after:

i. 200 years: **1.0822 × 10^(13)**

Doing a similar calculation we obtain a value of the trust that is $1000 \times 2^{2000/6} \doteq 2.2046 \times 10^{103}$

ii. 2000 years: **2.2046 × 10^(103)**

This example is a wonderful illustration of the ability of compounded interest to help accumulate money, over a long term.

15. Determine the time, t, required for an investment of $5000 to grow to $8500 at an interest rate of 7.5% per year, compounded quarterly:

Solution:

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The formula for compound interest investments is: $V(t) = P\left(1+\frac{i}{n}\right)^{nt}$, where V is the value of the investment at time t, P is the principal, or amount initially invested, i is the annual projected rate of interest (APR) and n is the number of times per year that interest is compounded.

The phrase "an investment of \$5000" tells us that P = 5000. The phrase "grow to \$8500" tells us that V = 8500. The sentence fragment "an interest rate of 7.5% per year, compounded quarterly" tells us that i = 0.075 and that n = 4. The phrase "Determine the time" tells us that it is a value of t that we must determine. Substituting the known values of P, V, i and n into the formula we obtain:

$$8500 = 5000\left(1+\frac{0.075}{4}\right)^{4t} \text{ (divide both sides by 5000) } \Leftrightarrow \frac{\overset{17}{\cancel{85}\,\cancel{00}}}{\underset{10}{\cancel{50}\,\cancel{00}}} = \frac{\cancel{5000}}{\cancel{5000}}(1.01875)^{4t}$$

$$\frac{17}{10} = (1.01875)^{4t} \text{ (take ln( ) on both sides) } \Leftrightarrow \ln(1.7) = \ln\left((1.01875)^{4t}\right)$$

$$\Leftrightarrow \ln(1.7) = 4t\ln(1.01875) \text{ (divide both sides by 4ln(1.01875)}$$

$$\Leftrightarrow \frac{\ln(1.7)}{4\ln(1.01875)} = \frac{\cancel{4}t\cancel{\ln(1.01875)}}{\cancel{4}\cancel{\ln(1.01875)}} \Leftrightarrow t = \frac{\ln(1.7)}{4\ln(1.01875)} \doteq 7.14$$

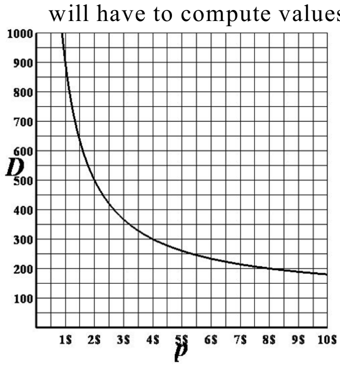
t = **7.14** years

Note that since interest rates are given as compounded quarterly, a more practical answer is really 7.25, at which time one will have a bit more than the exact \$8,500 requested.

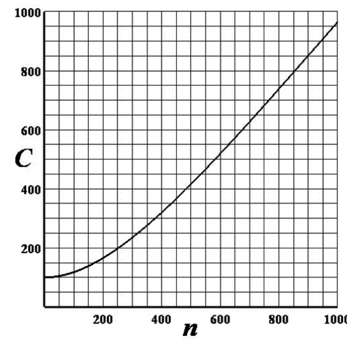
Appendix E - Common Assignment Problems

Paper Assignment #3 with Solutions

1. The AAA Widget Manufacturing Corporation has plotted a demand curve, with demand D in thousands of widgets as a function of price p, in \$ per widget, based on their last five years of experience as shown in graph A below. Note that graph A is drawn following conventions of mathematics with the independent variable p on the horizontal axis and the dependent variable D on the vertical axis, which is opposite to the orientation used in Economics classes. AAAWMC recently moved to a "Just In Time" widget manufacturing process, allowing them to produce exactly the number of widgets ordered, as they are ordered, with the cost of production C a function of the number of widgets, n, as shown in graph B below. The CEO of AAAWMC has just come into your office and demanded that you produce for him a graph of cost of production C versus price per widget p. This means that you will have to compute values of $C \circ D\ (p) = C(D(p))$ and plot them versus p.



A



B

Based on the given graphs (A and B), rounding to the value of the nearest vertex, compute the following values:

a. $C \circ D\ (1) = C(D(1))$, we note that when $p = 1$, then $D(1) = 900$, so $C(D(1)) = C(900)$, but when $n = 900$, we see that $C(900) = 850$, so $\boldsymbol{C \circ D\ (1) = 850}$
$C \circ D\ (4) = C(D(4)) = C(300) \doteq \mathbf{240}$
$C \circ D\ (8) = C(D(8)) = C(200) \doteq \mathbf{170}$
$C \circ D\ (10) = C(D(10)) \doteq C(180) \doteq \mathbf{160}$

b. Which of the four prices (\$1, \$4, \$8, \$10) generates the largest profit:
Computing profit involves first computing revenue (Demand × price), then subtracting total cost.
Thus, when $p = 1$, profit = $900 \times 1 - 850 = 50$.
When $p = 4$, profit = $300 \times 4 - 240 = 1200 - 240 = 960$.
When $p = 8$, profit = $200 \times 8 - 170 = 1600 - 170 = 1430$.
When $p = 10$, profit = $180 \times 10 - 160 = 1800 - 160 = 1640$.
Clearly looking at these four possibilities we see that the biggest profit comes when the price is set at \$10 and only 160 units are sold.

2. Given that $f(x) = (x - 1)^2$ and $g(x) = 2x + 3$, compute and then type in an appropriate numerical answer for each of the following function values:

Note: A useful trick. We rewrite each function replacing x with ( ), and then we place whatever we need into the empty brackets, ( ).

$f(x) = (x - 1)^2 \Leftrightarrow f(\ \ ) = ((\ \ ) - 1)^2$ and $g(x) = 2x + 3 \Leftrightarrow g(\ \ ) = 2(\ \ ) + 3$

a. $(f + g)(2) = f(2) + g(2) = ((\ 2\ ) - 1)^2 + 2(\ 2\ ) + 3 = 1 + 4 + 3 = \mathbf{8}$

b. $(f - g)(3) = f(3) - g(3) = ((\ ^3) - 1)^2 - (2(\ 3\ ) + 3) = 2^2 - (6 + 3) = 4 - 9 = \mathbf{-5}$

c. $(f \times g)(-1) = f(-1) \times g(-1) = ((\ -1\ ) - 1)^2 \times (2(\ -1\ ) + 3) = (-2)^2 \times (-2 + 3) = 4 \times 1 = \mathbf{4}$

d. $$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{((4)-1)^2}{2(4)+3} = \frac{3^2}{11} = \frac{9}{11}$$

e. $(f \circ g)(0) = f(\ g(0)\ ) = ((\ g(0)\ ) - 1)^2 = ((\ 2(0) + 3\ ) - 1)^2 = ((\ 3\ ) - 1)^2 = 2^2 = \mathbf{4}$

or $(f \circ g)(0) = f(\ g(0)\ ) = f((\ 2(0) + 3)) = f(3) = ((\ 3\ ) - 1)^2 = 2^2 = \mathbf{4}$

f. $(g \circ f)(1) = g(\ f(1)\ ) = 2(\ f(1)\ ) + 3 = 2\ (\ ((\ 1\ ) - 1)^2 + 3 = 2(0) + 3 = \mathbf{3}$

or $(g \circ f)(1) = g(\ f(1)\ ) = g(\ ((\ 1\ ) - 1)^2\ ) = g(\ (0)^2\ ) = g(0) = 2(0) + 3 = \mathbf{3}$

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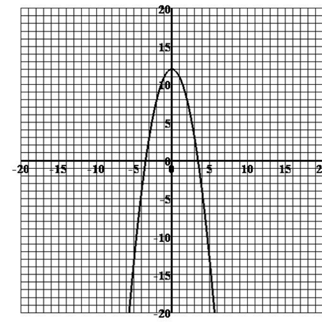
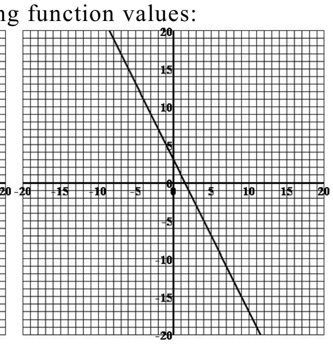
3. In the images below a graph of $f(x)$ is shown on the left and a graph of $g(x)$ is shown on the right. Using the graphs to compute values of f and g, compute and then type in an appropriate numerical answer for each of the following function values:





a. $(f+g)(2)=$ **7**

$(f+g)(2)=f(2)+g(2)=8+(-1)=7$

b. $(f-g)(3)=$ **6**

$(f-g)(3)=f(3)-g(3)=3-(-3)=6$

c. $(f\times g)(-1)=$ **55**

$(f\times g)(-1)=f(-1)\times g(-1)=11\times 5=55$

d. $\left(\frac{f}{g}\right)(4)=$ **1**

$$\left(\frac{f}{g}\right)(4)=\frac{f(4)}{g(4)}=\frac{-4}{-4}=1$$

e. $(f\circ g)(0)=$ **3**

$(f\circ g)(0)=f(\,g(0)\,)=f(\,3\,)=3$

f. $(g\circ f)(1)=$ **-19**

$(g\circ f)(1)=g(\,f(1)\,)=g\,(11)=-19$

4. Let $f(x)=\dfrac{1}{x-4}$ and $g(x)=3x+5$, then: a. $(f\circ g)(4)=-\dfrac{1}{9}$ and b. $(f\circ g)(x)=\dfrac{1}{3x+1}$

$$(f\circ g)(4)=f(\,g(4)\,)=f(3(4)+5)=f(17)=\frac{1}{(17)-4}=\frac{1}{13}$$

$$(f\circ g)(x)=f(g(x))=f(3x+5)=\frac{1}{(3x+5)-4}=\frac{1}{3x+1}$$

or $$(f\circ g)(x)=f(g(x))=\frac{1}{(g(x))-4}=\frac{1}{(3x+5)-4}=\frac{1}{3x+1}$$

5. Let $f(x)=\dfrac{1}{x-4}$ and $g(x)=\dfrac{4}{x}+4$, then: a. $(f\circ g)(x)=\dfrac{x}{4}$ and b. $(g\circ f)(x)=4(x-3)$

$$(f\circ g)(x)=f(g(x))=f\left(\frac{4}{x}+4\right)=\frac{1}{\left(\frac{4}{x}+4\right)-4}=\frac{1}{\frac{4}{x}+\cancel{4}-\cancel{4}}=\frac{1}{\left(\frac{4}{x}\right)}=\frac{x}{4}$$

or $$(f\circ g)(x)=f(g(x))=\frac{1}{(g(x))-4}=\frac{1}{\left(\frac{4}{x}+4\right)-4}=\frac{1}{\frac{4}{x}+\cancel{4}-\cancel{4}}=\frac{1}{\left(\frac{4}{x}\right)}=\frac{x}{4}$$

$$(g\circ f)(x)=g(f(x))=g\left(\frac{1}{x-4}\right)=\frac{4}{\left(\frac{1}{x-4}\right)}+4=4\left(\frac{x-4}{1}\right)+4=4x-16+4=4x-12=4(x-3)$$

$$(g\circ f)(x)=g(f(x))=\frac{4}{f(x)}+4=\frac{4}{\left(\frac{1}{x-4}\right)}+4=4\left(\frac{x-4}{1}\right)+4=4x-16+4=4x-12=4(x-3)$$

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6. Let $f(x) = 9x - 8$ and $g(x) = \dfrac{x+8}{9}$, then: a. $(f \circ g)(x) =$ and b. $(g \circ f)(x) =$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+8}{9}\right) = \cancel{9}\left(\frac{x+8}{\cancel{9}}\right) - 8 = x + \cancel{8} - \cancel{8} = x$$

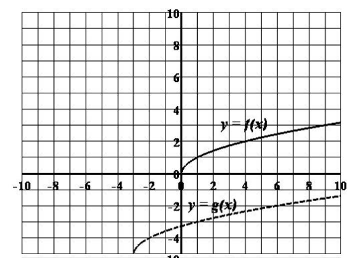
or $$(f \circ g)(x) = f(g(x)) = 9(g(x)) - 8 = \cancel{9}\left(\frac{x+8}{\cancel{9}}\right) - 8 = x + \cancel{8} - \cancel{8} = x$$

and $$(g \circ f)(x) = g(f(x)) = g(9x-8) = \frac{(9x-8)+8}{9} = \frac{9x - \cancel{8} + \cancel{8}}{9} = \frac{\cancel{9}x}{\cancel{9}} = x$$

or $$(g \circ f)(x) = g(f(x)) = \frac{(f(x))+8}{9} = \frac{(9x-8)+8}{9} = \frac{9x - \cancel{8} + \cancel{8}}{9} = \frac{\cancel{9}x}{\cancel{9}} = x$$

c. Thus $g(x)$ is called an **inverse** function of $f(x)$.

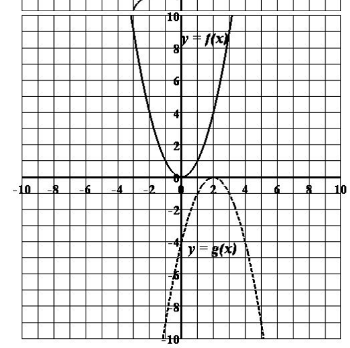
7. A graph of $f(x) = \sqrt{x}$ is sketched in solid black and on the same set of axes a graph of $g(x)$ is sketched in dashes.
Write an appropriate formula for the function $g(x)$, *i.e.*,



We note that the given graph of g is the same as that of f, except that g has been shifted (or translated) both horizontally, left 3 units, and vertically, down 5 units. Thus, $g(x) = \sqrt{x+3} - 5$

$g(x) = \sqrt{x+3} - 5$

8. A graph of $f(x) = x^2$ is sketched in solid black and a graph of $g(x)$ is sketched in dashes.
Write an appropriate formula for the function $g(x)$, *i.e.*,



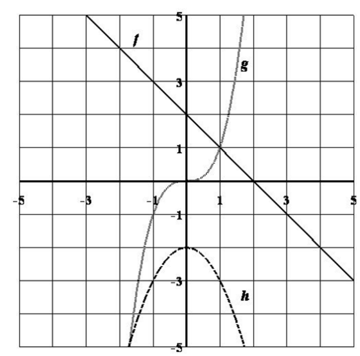
We note that the given graph of g is the same as that of f, except that g has been shifted (or translated) horizontally, right 2 units, and then flipped vertically about the x-axis. Thus, $g(x) = -(x-2)^2$.

$g(x) = \mathbf{-(x - 2)^2}$

9. Given $f(x) = x^2$, after performing the following transformations: shift 53 units to the right
shifting 53 units to the right corresponds to the function $h(x) = (x - 53)^2$
and then upwards 36 units, the formula for the new function,
shifting 36 units upwards corresponds to the function $g(x) = (x - 53)^2 + 36$

$g(x) = \mathbf{(x - 53)^2 + 36}$

10. Match the functions shown in the graph below (f, g, and h) with the formulae below:



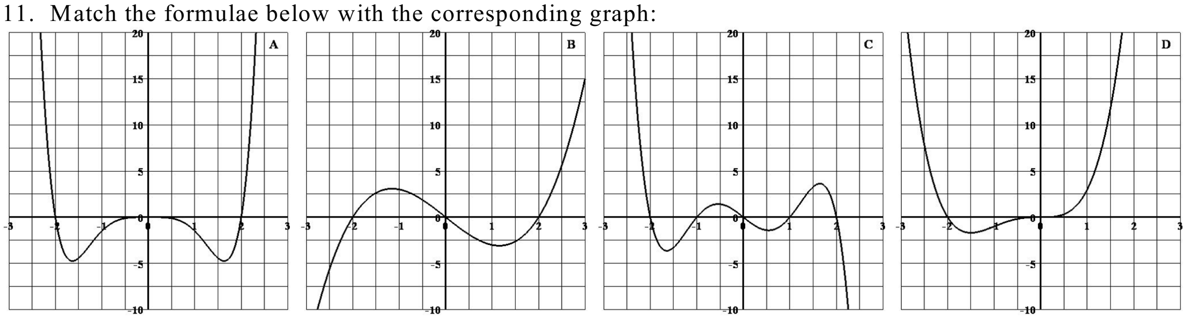
a. $\boldsymbol{h(x)} = -x^2 - 2$ (Clearly this is a quadratic with negative leading coefficient. Thus, the graph is an upside down parabola, namely h.

b. $\boldsymbol{g(x)} = x^3$ (Clearly this given graph is a cubic power function, namely g.)

c. $\boldsymbol{f(x)} = -x + 2$ (Clearly this is a linear function with $m = -1$, and $b = 2$, which matches the graph of f.

Appendix E - Common Assignment Problems

11. Match the formulae below with the corresponding graph:



A B C D

B = $x(x^2 - 4)$, **D?** = $x^4 + 2x^2$, **A** = $\dfrac{x^6}{2} - 2x^4$, **C** = $-x^5 + 5x^3 - 4x$

There was an error in this problem. We note that the first formula would give rise to a cubic or degree three polynomial, with x-intercepts at $x = -2, 0, 2$. The leading coefficient is positive (1), so at the edges this graph should behave like the power function x^3. Thus, this first function is associated with graph B.

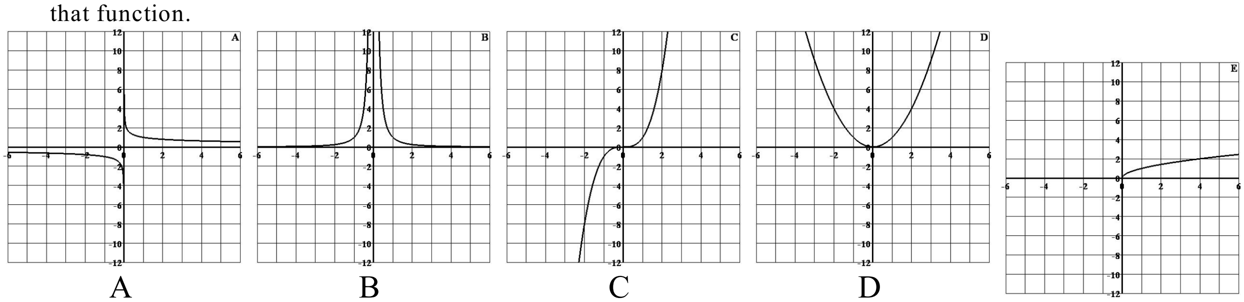
Skipping past the second formula for the moment and looking at the third one, we note that we could factor out $(½)x^4$, leaving $(x^2 - 4)$, which in turn factors as $(x - 2)(x + 2)$, just like in the first formula. This means that a graph of this degree 6 polynomial has three x-intercepts, $x = -2, 0, 2$, just like the first formula. However, the x^4 factor influences that middle intercept to create a U-shaped intersection that touches the x-axis, but does not cross it. A graph of this third function will also be symmetric about the y-axis, and at the edges it will resemble a graph of the power function x^6. This means that the third formula is associated with graph A.

The fourth and final formula is a degree 5 (odd number) polynomial with a negative leading coefficient. The odd degree and negative leading coefficient tell me that the edge behaviour of this function will resemble that of the power function $-x^5$. We also notice that we could factor out an x from this formula, meaning that $x = 0$ should be an x-intercept of this function. This means that only graph C could fit this formula. To test it further we note that $x = -2, -1, 0, 1, 2$ are x-intercepts on this graph. This should mean that if we substitute any of these values into the formula we should obtain 0 as the value of the function. We already know that this is true for $x = 0$. Let $f(x) = -x^5 + 5x^3 - 4x$. Then we compute as follows: $f(-2) = -(-2)^5 + 5(-2)^3 - 4(-2) = 32 - 40 + 8 = 0$;
$f(-1) = -(-1)^5 + 5(-1)^3 - 4(-1) = 1 - 5 + 4 = 0$; $f(1) = -(1)^5 + 5(1)^3 - 4(1) = -1 + 5 - 4 = 0$; $f(2) = -(2)^5 + 5(2)^3 - 4(2) = -32 + 40 - 8 = 0$.
Alternatively, instead of doing this arithmetic, we could have done the algebra of factoring $f(x)$ as follows:
$f(x) = -x^5 + 5x^3 - 4x = -x(x^4 - 5x^2 + 4) = -x(x^2 - 4)(x^2 - 1) = -x(x + 2)(x - 2)(x + 1)(x - 1)$. Thus we see the zeroes of this function, and they match the x-intercepts of graph C.

By process of elimination, graph D should match the second formula. Sadly there was a typing error, and instead of $x^4 + 2x^3 = x^3(x + 2)$, which does match graph D, what was typed is $x^4 + 2x^2 = x^2(x^2 + 2)$, which has only one x-intercept, U shaped, at $x = 0$.

Note: In correction of this problem, basically only responses to the first, third and fourth formulae count for marks.

12. Given the five graphs below, A., B., C., D., and E., indicate beside each of the five functions below, which graph corresponds to that function.



A B C D E

x^2: **D** $x^{½}$: **E** x^{-2}: **B** x^3: **C** $x^{-⅓}$: **A**

We note that the five given formulae are simple power functions, *i.e.*, x^a, where a is some constant. The question then is which power function corresponds to which graph.

The graph of x^2 is well known. Called a parabola, its shape is often referred to as a "U", and we note that the graph D is the only one that matches the expected shape.

Appendix E - Common Assignment Problems

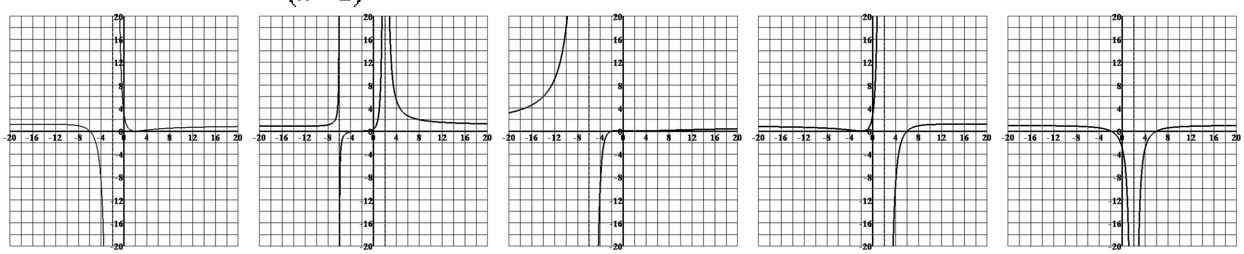
The graph of $x^{½} = \sqrt{x}$ is also well known. The graph should start at $x = 0$, and rise to the right but be entirely concave down. This function is not defined for $x < 0$, so there is no graph to the left of $x = 0$. The only graph matching these characteristics is graph E.

The graph of x^{-2} should be symmetric about the y-axis (since for example $x = -5$ and $x = 5$ would yield the same y-value, 1/25). Second, at the edges, where the absolute value of x gets very large, the values of $x^{-2} = 1/x^2$ get very small, close to 0. That is, a graph of this function should have $y = 0$ as a horizontal asymptote at both edges. Further, when x is close to 0, the value of $1/x^2$ would be very large, so a graph of this function should have a vertical asymptote, heading upwards towards ∞ on both sides of the vertical line $x = 0$. The only graph that matches these characteristics is B.

The graph of x^3 should also be known. It should pass through the origin. It should also exhibit symmetry about the origin. That is, for example the y-value of $x = 1$ and $x = -1$ should have the same absolute value, just be of opposite sign. The only graph that matches both of these characteristics is C.

The graph of $x^{-⅓} = 1/x^{⅓}$ should exhibit symmetry about the origin just like the previous function, because of the odd parity of the 3. However, when x is very large in absolute value, *i.e.*, at the edges of the x-axis, the denominator would be large, and the fraction would be close to 0. That is, at the edges of the x-axis, the graph of this function would approach the horizontal asymptote $y = 0$. Further, when x is close to 0, the value of $1/x^{⅓}$ would be very large in absolute value because $x^{⅓}$ would be close to 0. The only graph matching all of these characteristics, the only graph left, is graph A.

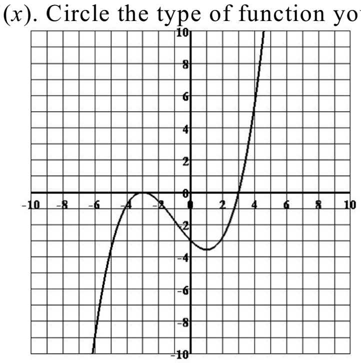
13. Given that $f(x) = \dfrac{(x+2)^2(x-6)}{(x-2)^3}$, from the five graphs given below, select which represents f. That is, the graph of f is: **D**



A B C D E

We note that the denominator is $(x - 2)^3$, and there is no factor $(x - 2)$ in the numerator. This indicates that this graph should, in absolute value, be very large, for values of x close to 2. This is because when x is close to 2, then $(x - 2)$ should be close to 0. Because the power 3 of the factor in the denominator is odd, the denominator changes sign as x passes from the left of 2 to the right of 2. This means that on one side of $x = 2$ the graph should head upwards towards $+\infty$ and on the other side the graph should head downwards towards $-\infty$. We also note the factor $(x + 2)^2$ in the numerator, and no factor $(x + 2)$ in the denominator. This means that $x = -2$ is a zero of the function, and a graph of the function should have an x-intercept, U shaped, at $x = -2$. Finally we note the factor $(x - 6)$ in the numerator and no such factor in the denominator. This means that $x = 6$ is a zero of the function, and a graph of the function should have an x-intercept, cutting the x-axis at $x = 6$ like an oblique line. The only graph matching all of these characteristics is D.

14. The graph below is of an unknown function $f(x)$. Circle the type of function you think $f(x)$ is.



a. linear power exponential logarithmic **polynomial** rational trigonometric

Clearly this function is not linear (not a straight line graph), nor does it resemble any of the power function graphs (either passing through the point (0,0) or have the line $x = 0$ as a vertical asymptote). It does not repeat in the fashion of a trigonometric function graph, all of which are periodic. It does not have the line $y = 0$ as a horizontal asymptote at one edge and head towards $\pm\infty$ at the other edge, so it is not the graph of an exponential function. If it were the graph of a logarithmic function, it should only be defined on one side of some vertical line, $x = a$, and that vertical line should serve as a vertical asymptote to the graph on that side. The given graph

Appendix E - Common Assignment Problems

has no vertical asymptote, so this cannot be the graph of a logarithmic function. This leaves two possibilities. Since all polynomials are technically rational functions, just with denominator polynomial 1, the safest answer in some sense is rational. However, looking at the graph, and at the next question, it does have the typical shape of a cubic or degree three polynomial. Hence, the best answer is polynomial function.

b. Write a formula for $f(x)$: $(1/9)(x + 3)^2(x - 3)$

Hint: Use as simple a formula as you can - don't over complicate your solution.

We note the U-shaped x-intercept at $x = -3$, indicating a factor $(x + 3)^n$, where n is an even positive integer. The simplest choice for n is 2. We note the x-intercept at $x = 3$, where the graph cuts the x-axis like an oblique line, indicating a factor of $(x - 3)$. Thus $f(x) = a(x + 3)^2(x - 3)$ (formula #1), for some constant a. We can solve for a if we can observe any single point on the graph of f except an x-intercept. The point on the given graph with coordinates easiest to observe is the point (0,-3). This means that $f(0) = -3$. Substituting this information into formula #1 we obtain: $-3 = f(0) = a((0) + 3)^2((0) - 3) \Leftrightarrow -3 = a(9 \times (-3)) \Leftrightarrow a = 1/9$. Thus, $f(x) = (1/9)(x + 3)^2(x - 3)$

15. In class and in your text you have seen a variety of graphs of power functions where the shape of the graph depends upon the value of the power. That is, if $f(x) = x^p$, then changing p changes the shape of a graph of $f(x)$. Without computing anything, but just by observing the graph of the power function x^p for different values of p, indicate whether each of the following statements is true or false.

a. **False** All power functions are increasing.

We have observed the graph of x^{-2}, which for $0 < x$ has a graph which is decreasing.

b. **False/True** The power function x^{-1} is decreasing everywhere it is defined.

It is clear that on the interval $(-\infty,0)$ this function has a graph that is decreasing. Similarly on the interval $(0,\infty)$ this function also has a graph that is decreasing. Further, the function is not defined at $x = 0$. Thus, these two separate pieces of the graph of this function are both decreasing. In this way it seems that the correct answer is **True**.

Strictly speaking the definitions of increasing or decreasing for a function f are made for intervals. That is, a function is said to be increasing (or decreasing) on an interval (a, b) if for every pair of x's, x_1 and x_2 such that $x_1, x_2 \in (a, b)$, with $x_1 < x_2$, then $f(x_1) < f(x_2)$ (or $f(x_1) > f(x_2)$). Thus, the statement should really have been: The power function x^{-1} is decreasing on every interval where it is defined.

However, the way that the statement currently stands there is an ambiguity. For example, we know that $-1 < 1$ and $f(-1) = 1/(-1) = -1 < f(1) = 1/1 = 1$, which is the opposite of the inequality required for a decreasing function. Viewed this way it makes it look like the statement is False. Of course all of the difficulty comes about because of the vertical asymptote that this function has at $x = 0$. As we shall see later in the course, difficulties because of graphical features such as vertical asymptotes are part of what we study in Calculus I.

c. **False** All power functions with $p > 1$ are concave up.

Looking at the left side of the graph of x^3 ($p=3>1$) we note immediately that it is concave down.

d. **False** All power functions are only defined for positive values of the input (x here).

Looking at the graph of x^2, we note that it is defined for all values of x.

e. **False** All power functions with $0 < p < 1$ are concave down, *i.e.*, their growth rate slows down on every interval.

The graph of $x^{1/3}$, for $x < 0$, is in fact concave up, contradicting this statement.

Appendix E - Common Assignment Problems

Paper Assignment #4 with Solutions

1. Sony BMG Music Corporation has found, based on data from 5 years of sales, that the following is a realistic model for profit, P (in \$10,000,000's), as a function of volume of sales, x (in millions of CD's): $P(x)=\dfrac{9}{1+e^{-x}}-2$. Your boss, CEO Rolf Schmidt-Holtz has to report to a board meeting on current sales and sales growth and asks you to give him data in one half-hour. You note that last quarter your sales were 2 million CD's. To estimate the instantaneous rate of change of profit, when sales are at 2 million CD's **you compute the average rate of change for sales** beginning with 2 million CD's and going to (give answers accurate to four decimal places and in dollars per CD):
   (i) 3 million CD's: **0.646** (ii) 2.1 million CD's: **0.909** (iii) 2.001 million CD's: **0.9**
   Based on these calculations, when you report to your boss what is your best estimate for instantaneous rate of change of profit (\$/CD) when sales are 2 million CD's: **0.9**

Solutions:

We observe that $P(2)=\dfrac{9}{1+e^{-2}}-2 \doteq 5.9272$; $P(3)=\dfrac{9}{1+e^{-3}}-2 \doteq 6.5732$; $P(2.1)=\dfrac{9}{1+e^{-2.1}}-2 \doteq 6.0181$

and $P(2.001)=\dfrac{9}{1+e^{-2.001}}-2 \doteq 5.9281$

Using these calculations we calculate the average rate of change over each of the requested intervals, noting that computing the average rate of change is the same as computing slope of a secant line.

Interval (2,3): $\dfrac{P(3)-P(2)}{3-2} \doteq \dfrac{6.5732-5.9272}{1}=0.646$

Interval (2,2.1): $\dfrac{P(2.1)-P(2)}{2.1-2} \doteq \dfrac{6.0181-5.9272}{0.1}=0.909$

Interval (2,2.001): $\dfrac{P(2.001)-P(2)}{2.001-2} \doteq \dfrac{5.9281-5.9272}{0.001}=0.9$

The values of these average rates of change seem to be heading towards 0.9, or perhaps a touch lower. Our best estimate should be just the one based on the shortest interval, which is the last one, 0.9,

2. You are working as assistant to a major movie producer from Hollywood, but mainly you look after pre-release film publicity. You are an old hand at publicity, having worked in several production companies before. Based on previous experience you have developed a model for the effective expenditure of money on radio and television advertising in the last ten days prior to the release of the film. Money spent, M (in \$100,000) is a function of the number of days remaining prior to the release of the film, n ($n \le 10$), is given as: $M(n)=\dfrac{n^3-48n+1500}{200}$. The producer, your boss, comes in and asks you to prepare a report on the funding needs for Radio and Television advertising in the last ten days prior to release, highlighting the rate at which spending goes up or down each day. In particular, he seems interested in the rate at which spending will change four days from the end. To estimate this, using your formula, calculate the average rate of change for each of the following intervals (give answers accurate to four decimal places):
   n on [0, 1]: **-0.235** n on [1, 2]: **-0.205** n on [2, 3]: **-0.145** n on [3, 4]: **-0.055**
   n on [4, 5]: **0.065** n on [5, 6]: **0.215** n on [6, 7]: **0.395** n on [7, 8]: **0.605**
   n on [8, 9]: **0.845** n on [9, 10]: **1.115** n on [5, 7]: **0.305**
   Your best estimate for the instantaneous rate of change at n = 4: **0.005**
   Why was the manager interested in the fourth day from the end: **circle the best answer**
   (no reason) **<u>(spending starts increasing after that day)</u>**
   (spending starts decreasing after that day) (that day is the peak of expenditures)

We observe that: $M(0)$ = 7.5; $M(1)$ = 7.265; $M(2)$ = 7.06; $M(3)$ = 6.915; $M(4)$ = 6.86; $M(5)$ = 6.925; $M(6)$ = 7.14; $M(7)$ = 7.535; $M(8)$ = 8.14; $M(9)$ = 8.985; $M(10)$ = 10.1

Using the above computations we compute average rates of change for each of the intervals:
[0,1]: -0.235; [1,2]: -0.205; [2,3]: -0.145; [3,4]: -0.055; [4,5]: 0.065; [5,6]: 0.215; [6,7]: 0.395; [7,8]: 0.605; [8,9]: 0.845; [9,10]: 1.115; [5,7]: 0.305 and [3,5]: 0.005

The best estimate for the instantaneous rate of change at n = 4 is arrived at by either averaging the two average rates of change for [3,4] and [4,5], or what amounts to the same thing, calculating the average rate of change for [3,5].

Appendix E - Common Assignment Problems

We note that the average rate of change on intervals prior to and including [3,4] is negative, but on intervals after and including [4,5] is positive. That is, function values were decreasing in value until $n = 4$ and increasing after $n = 4$. Thus, the manager is interested in what is happening at day $n = 4$ because (spending starts increasing after that day).

3. The function $f(t) = 5t^2 + 10t$ represents the distance of a car from an intersection (measured in metres) t seconds after leaving the intersection. Calculate the instantaneous velocity of the car 10 seconds after it has left the intersection: **110 (You can use a calculator to compute velocity.)**
   Remember this technique for determining instantaneous velocity or instantaneous rate of change. Later we will use the same method to determine the derivatives of function such as $f(t)$.

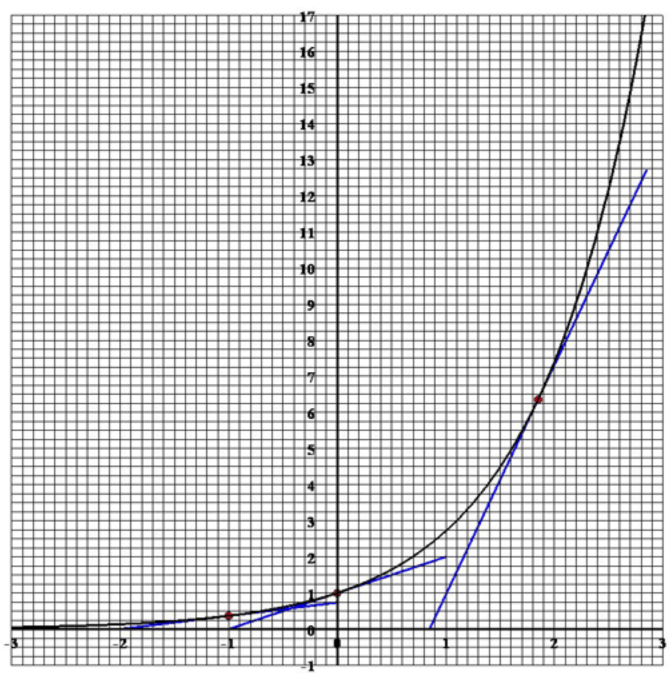
We make a table of values of $f(t)$ where t gets progressively closer and closer to 10. Then we use those values to compute average velocities on intervals of shorter and shorter width on either side of 10. Hopefully we see a pattern in the values of the average velocities so that we can predict what the instantaneous velocity it that they are headed towards.

t	9.9		9.99		9.999		10		10.001		10.01		10.1
$f(t)$	589.05		598.9005		599.890005		600		600.110005		601.1005		611.05
avg. vel.	109.5		109.95		109.995				110.005		110.05		110.5

Typical average velocity calculation, interval [9.9,10]: $\dfrac{f(10)-f(9.9)}{10-9.9} = \dfrac{600-589.05}{0.1} = 10.95 \times 10 = 109.5$

We note that the average velocities for intervals with 10 as the right edge or higher number are numbers less than 110, but seem to be getting closer and closer to 110. On the other side, the average velocities for intervals with 10 as the left edge or lower number are numbers larger than 110, but seem to be getting closer and closer to 110. Thus, our estimate for instantaneous velocity at $t = 10$ is 110 (m/sec). **(Note: whoever is driving this car is travelling dangerously fast since 110 (m/sec) is 396 (km/hr).)**

4. The graph on the right is of the function $f(x) = e^x$. It also shows tangent lines to this curve at $x = -1$, 0, and 1. Use your calculator to compute slopes of secant lines until you can estimate the slopes of the drawn tangent lines to 4 decimal place accuracy.
   Slope of tangent line at $x = -1$ or $f'(-1)$: _____
   Slope of tangent line at $x = 0$ or $f'(0)$: _____
   Slope of tangent line at $x = 1.85$ or $f'(1.85)$: _____



Note: You must keep computing better and better approximations until you can see 4 decimal places stabilize. This may involve as many as 7 or 8 approximations each time. If your calculator is programmable, or you know how to use a spreadsheet (which you should), you can automate and speed up this process greatly.
Just like the previous problem, we generate tables of values of the function, and then of slopes of secant lines, until we are confident that we can see, at least to four decimal places, where the slopes of secant lines appear to be headed, and thus estimate the slopes of tangent lines.

x	-1.001		-1.0001		-1.00001		-1		-0.99999		-0.9999		-0.999
$f(x)$	0.367512		0.367843		0.367876		0.367879		0.367883		0.367916		0.368248
m_{sec}	0.367696		0.367861		0.367878				0.367881		0.367898		0.368063

We can see that the two smallest width intervals, [-1.00001,-1] and [-1,0.99999] give rise to secant line slopes that are the same when rounded to four decimal places, 0.3679. Thus, based on the above table we believe that for $x = -1$, $m_{tan} \doteq 0.3679$ (rounded to four decimal places).

Appendix E - Common Assignment Problems

x	-0.001		-0.0001		-0.00001		0		0.00001		0.0001		0.001
$f(x)$	0.999000		0.999900		0.999990		1		1.000001		1.000100		1.001000
$m_{\sec}$	0.995		0.99995		0.999995				1.000005		1.00005		1.0005

We can see that the two smallest width intervals, [-0.00001,0] and [0,0.00001] give rise to secant line slopes that are the same when rounded to four decimal places, 1.0000. Thus, based on the above table we believe that for $x = 0$, $m_{\tan} \doteq 1.0000$ (rounded to four decimal places).

x	1.849		1.8499		1.84999		1.85		1.85001		1.8501		1.851
$f(x)$	6.353463		6.359184		6.359756		6.35982		6.359883		6.360456		6.366183
$m_{\sec}$	6.356641		6.359502		6.359788				6.359851		6.360138		6.363

We can see that the two smallest width intervals, [1.84999,1] and [1,1.85001] give rise to secant line slopes that are the same when rounded to four decimal places, 6.3598. Thus, based on the above table we believe that for $x = 1$, $m_{\tan} \doteq 6.3598$ (rounded to four decimal places).

Note: There is an interesting observation to be made here. Our estimate of the slope of the tangent at the three *x*-values are all in fact the same as the value of the function at that same *x*-value. We know from the previous example that this is not the case for other functions. Why is it true here?

5. The data in the table below define S, the "single day take" (amount of money taken in during one day at theatres showing a particular film) (measured in $1,000,000) as a function of the number of days after commercial release, n.

n	0	1	2	3	4	5
S	14.0	12.6	12.5	12.8	13.3	14.3

a. Let P be the point (3, 12.8). Determine the slopes of the secant lines PQ (average rate of change of single day take), when Q is the point of the graph with S coordinates:

n	0	1	2	4	5
S	**-0.4**	**0.1**	**0.3**	**0.5**	**0.75**

A typical calculation here is, for the interval [0,3]: $\dfrac{12.8-14.0}{3-0}=\dfrac{-1.2}{3}=-0.4$

b. Draw the graph of the function for yourself and estimate the value of the slope of the tangent line at P (instantaneous rate of change of single day take, or "marginal single day take"): **0.4**

Averaging our slope of secant line from the intervals [2,3] and [3,4] we obtain 0.4

6. The point P (4,4) lies on the curve $f(x) = \sqrt{x}+2$. If Q is the point $\left(x,\sqrt{x}+2\right)$, determine the slope of the secant line PQ, for the following values of x:

a. If $x = 4.1$, then the slope of PQ is: **0.248457**
b. If $x = 4.01$, then the slope of PQ is: **0.249844**
c. If $x = 3.9$, then the slope of PQ is: **0.251582**
d. If $x = 3.99$, then the slope of PQ is: **0.250156**
e. Based on the results above, guess the slope of the tangent line to this curve at P (4,4): **0.25**

This problem is similar to the previous ones, just focussed on $x = 4$ and $f(x) = \sqrt{x}+2$. We make a table of values of the function, and of slopes of secant lines and then estimate the slope of the tangent line based on our two best slopes of secant lines coming from the smallest width intervals on either side of $x = 4$.

x	3.9		3.99		4		4.01		4.1
$f(x)$	3.974842		3.997498		4		4.002498		4.024846
$m_{\sec}$	0.251582		0.250156				0.249844		0.248457

Appendix E - Common Assignment Problems

We can see that the two smallest width intervals, [3.99,4] and [4,4.01] give rise to secant line slopes that are 0.250156 and 0.249844 respectively. Averaging the two of these we obtain our best estimate for the slope of a tangent line at $x = 4$, $m_{\tan} \doteq 0.25$.

7. The point P (0.5,8) lies on the curve $f(x) = \frac{4}{x}$. If Q is the point $\left(x, \frac{4}{x}\right)$, determine the slope of the secant line PQ, for the following values of x:
   a. If $x = 0.6$, then the slope of PQ is: **-13.3333**
   b. If $x = 0.51$, then the slope of PQ is: **-15.6863**
   c. If $x = 0.4$, then the slope of PQ is: **-20**
   d. If $x = 0.49$, then the slope of PQ is: **-16.3265**
   e. Based on the results above, guess the slope of the tangent line to this curve at P (0.5,8): **-16.0064**

This problem is similar to the previous one, just focussed on $x = 0.5$ and $f(x) = \frac{4}{x}$. We make a table of values of the function, and of slopes of secant lines and then estimate the slope of the tangent line based on our two best slopes of secant lines coming from the smallest width intervals on either side of $x = 0.5$.

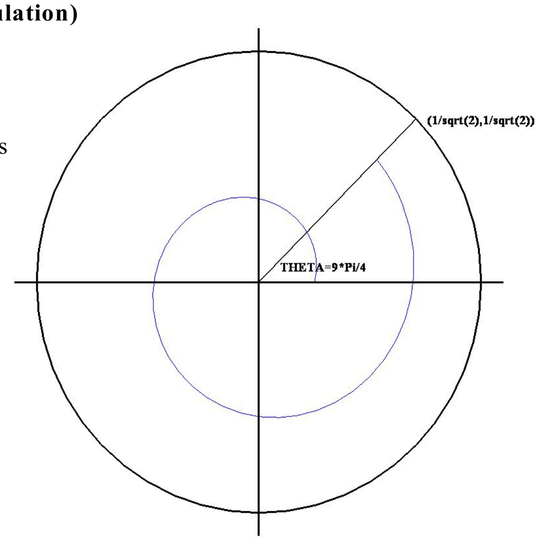
x	0.4		0.49		0.5		0.51		0.6
$f(x)$	10		8.163265		8		7.843137		6.666667
$m_{\sec}$	-20		-16.3265				-15.6863		-13.3333

We can see that the two smallest width intervals, [0.49,0.5] and [0.5,0.51] give rise to secant line slopes that are -16.3265 and -15.6863 respectively. Averaging the two of these we obtain our best estimate for the slope of a tangent line at $x = 0.5$, $m_{\tan} \doteq -16.0064$.

8. If $\theta = \frac{9\pi}{4}$, then provide answers **as fractions (not decimal numbers)** for each of the following. Notice that for the first two expressions the denominator is already included.

   sin(θ): **1 / $\sqrt{2}$** ; cos(θ): **1 / $\sqrt{2}$** ; tan(θ): **1 / 1**; sec(θ): **$\sqrt{2}$ / 1**

   (Suggestion: draw a unit circle and illustrate the angle θ as a guide for calculation)



First we note that $\frac{9\pi}{4} = \frac{8\pi}{4} + \frac{\pi}{4} = 2\pi + \frac{\pi}{4}$. This means that the terminal arm of the angle θ lies on top of the terminal arm of the angle $\frac{\pi}{4}$. We know that the coordinates of points on the unit circle are (cos(θ), sin(θ)) so that we can just read off the coordinates to obtain values of sine and cosine. Further, since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$, we use values of sin(θ) and cos(θ).

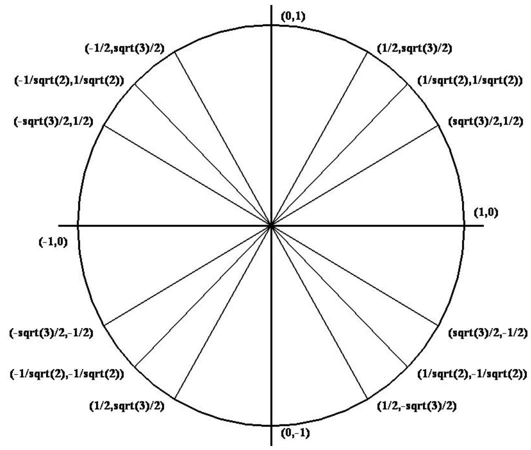
9. Evaluate the expressions below. Your answers cannot contain any function names, just arithmetic expressions, **not decimal numbers**.

   $\sin\left(-\frac{\pi}{6}\right)$: **-1 / 2** $\cos\left(\frac{3\pi}{2}\right)$: **0 / 1** tan(-π): **0 / (-1)**

   $\cot\left(\frac{3\pi}{4}\right)$: **-1** $\sec\left(\frac{\pi}{4}\right)$: **$\sqrt{2}$ / 1** $\csc\left(-\frac{\pi}{4}\right)$: **- $\sqrt{2}$ / 1**

   (Suggestion: draw a unit circle and illustrate the angles as guides for calculation)

Appendix E - Common Assignment Problems



Since the angle $-\frac{\pi}{6}$ is co-terminal with the angle $\frac{11\pi}{6}$, we read the answer for $\sin\left(-\frac{\pi}{6}\right)$ by looking at the *y*-coordinate of $\frac{11\pi}{6}$. The angle $\frac{3\pi}{2}$ has its terminal arm on the negative *y*-axis so we can read the answer for $\cos\left(\frac{3\pi}{2}\right)$ by looking at the *y*-coordinate there.

The angle $-\pi$ is co-terminal with the angle π. Thus, to determine $\tan(-\pi)$ we just take the ratio of *y*-coordinate over *x*-coordinate from the angle π.

The angle $\frac{3\pi}{4}$ is on our unit circle chart, so we can read the values of $\cos\left(\frac{3\pi}{4}\right)$ and $\sin\left(\frac{3\pi}{4}\right)$, and then compute $\cot\left(\frac{3\pi}{4}\right) = \frac{\cos\left(\frac{3\pi}{4}\right)}{\sin\left(\frac{3\pi}{4}\right)}$.

10. Given that $f(x) = -15\sin(2x + 7)$, determine the information below. Your answers may contain the letter π, but should **not be decimal** approximations.
    amplitude of f: **15** period of f: π phase shift of f: **-7/2**

We begin by rewriting $f(x)$: $f(x) = -15\sin(2x + 7) = -15\sin(2(x + 7/2))$. Rewriting $f(x)$ in this fashion allows us to process the steps used in transforming the function $\sin(x)$ into $f(x)$ in a simple order:

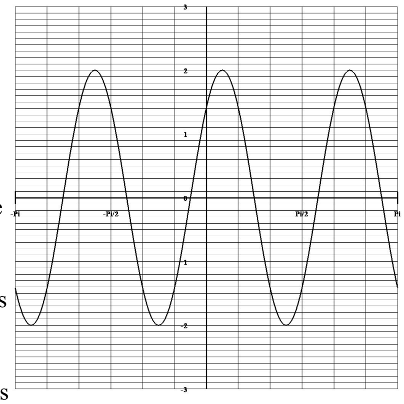
i) $x \to x + (7/2)$: this addition of 7/2, or rather subtraction of (-7/2), causes a horizontal shift to the left of (7/2) units. This gives us the answer to the question of phase shift, namely that phase shift is -7/2. To see this clearly we note that $\sin(0) = \sin(\pi) = \sin(2\pi) = 0$. We note that for $\sin(x + 7/2)$, these zeroes are achieved at $x = -7/2$, $\pi - 7/2$, $2\pi - 7/2$. That is: $\sin((-7/2)+7/2) = \sin(0)$; $\sin((\pi - 7/2) + 7/2) = \sin(\pi)$; $\sin((2\pi - 7/2) + 7/2) = \sin(2\pi)$. This means that the function has been shifted 7/2 units to the left.

ii) $x + (7/2) \to 2(x + (7/2))$: this multiplication of the independent variable by 2 causes a horizontal contraction, by a factor of 2, towards $x = -(7/2)$. Now we know that the period of $\sin(x)$ is 2π, so the period of $f(x)$ is a contraction of that, namely, $(2\pi)/2 = \pi$. To see this clearly we note that the function $\sin(x + 7/2)$ was observed above to run through a period passing from $x = -7/2$ to $x = -7/2 + 2\pi$, a period of length 2π. But we note that $\sin(2(x + 7/2))$ runs through this same period from $x = -7/2$ to $x = -7/2 + \pi$, a period of length π. That is: $\sin(2((-7/2) + 7/2)) = \sin(2(0)) = \sin(0)$; $\sin(2((-7/2+\pi/2) + 7/2)) = \sin(2(\pi/2)) = \sin(\pi)$; $\sin(2((-7/2+\pi) + 7/2)) = \sin(2(\pi)) = \sin(2\pi)$.

iii) $\sin(2(x + (7/2))) \to 15\sin(2(x + (7/2)))$: this multiplication of the dependent variable by 15 causes a vertical dilation (stretching), by a factor of 15, away from $y = 0$. We know that the amplitude of $\sin(x)$, $\sin(x + 7/2)$, $\sin(2(x + 7/2))$ were all the same, 1, seen as the gap between $1 = \sin(\pi/2)$ and $-1 = \sin(3\pi/2)$. But now that gap is between
$15 = 15\sin(2((-7/2 + \pi/4) + 7/2)) = 15\sin(2(\pi/4)) = 15\sin(\pi/2)$ and
$-15 = 15\sin(2((-7/2 + 3\pi/4) + 7/2)) = 15\sin(2(3\pi/4)) = 15\sin(3\pi/2)$. Thus, the amplitude has been stretched by this factor of 15 from 1 to 15.

iv) $15\sin(2(x + (7/2))) \to -15\sin(2(x + (7/2)))$: this multiplication of the dependent variable by -1 causes a vertical flip about the horizontal line, $y = 0$ (the *x*-axis). We can see this because for any given *x*, the point previously graphed was $(x, 15\sin(2(x + (7/2))))$ where now it is $(x, -15\sin(2(x + (7/2))))$, which has the same *x*-coordinate and a *y*-coordinate with the same absolute value, but opposite sign (hence the mirror image of the original point, where the *x*-axis is the mirror).

Appendix E - Common Assignment Problems



11. The graph on the right shows a periodic function, $f(x)$.
Determine the information requested below. Your answers may contain π and **should not be decimal approximations**.
period of f: **2π/3** amplitude of f: **2**
Write a possible formula for this function: $f(x)$ = **2sin(3(x + π/12))**

We note that the graph appears to be a modified version of either sine or cosine (we can make it be either one - we will choose sine, you can try doing it for cosine). First we note that if this is the sine function, then the zero that sin(x) has at the origin (sin(0) = 0), has been shifted left to what appears to be -π/12. Thus, we imagine that a first step in transforming sin(x) into the given function would be the function sin(x + π/12). Next we note that the period of this function appears to be 2π/3, which is a horizontal contraction of the period of sin(x) and sin(x + π/12) by a factor of 3, so our next step in transforming sin(x) into the given function would be the function sin(3(x + π/12)). Finally, where sin(x) varies vertically between -1 and 1, the given function varies between -2 and 2, which is a vertical dilation of the amplitude by a factor of 2. Thus, the given function must be 2sin(3(x + π/12)).

12. Determine the value of the limit $\lim_{x\to 6} 5(7x+7)^3$ = **5(49)³**

$$\lim_{x\to 6} 5(7x+7)^3 = 5\lim_{x\to 6}(7x+7)^3 \quad \text{(limit of a constant times a function is the constant times the limit)}$$

$$= 5\left(\lim_{x\to 6}(7x+7)\right)^3 \quad \text{(limit of a power of a function is the power of the limit of the function)}$$

$$= 5\left(\left(\lim_{x\to 6} 7x\right)+\left(\lim_{x\to 6} 7\right)\right)^3 \quad \text{(limit of a sum of functions is the sum of the limits of the functions)}$$

$$= 5\left(7\left(\lim_{x\to 6} x\right)+7\right)^3 \quad \text{(limit of a constant times a function is the constant times the limit \& limit of a constant is the constant)}$$

$$= 5\left(7(6)+7\right)^3 \quad \text{(as } x \text{ goes to 6, } x \text{ goes to 6)}$$

$$= 5(49)^3$$

13. Determine the value of the limit $\lim_{x\to -1} \frac{7x^2-5x+7}{x-8}$ = **-(19/9)**

$$\lim_{x\to -1} \frac{7x^2-5x+7}{x-8} = \frac{\lim_{x\to -1} 7x^2-5x+7}{\lim_{x\to -1} x-8} \quad \text{(limit of a quotient is the quotient of the limits)}$$

$$= \frac{\left(\lim_{x\to -1} 7x^2\right)-\left(\lim_{x\to -1} 5x\right)+\left(\lim_{x\to -1} 7\right)}{\left(\lim_{x\to -1} x\right)-\left(\lim_{x\to -1} 8\right)} \quad \text{(limit of a sum or difference is a sum or difference of limits)}$$

$$= \frac{7\left(\lim_{x\to -1} x^2\right)-5\left(\lim_{x\to -1} x\right)+7}{-1-8} \quad \text{(limit of a constant times a function is the constant times the function \& limit of a constant is the constant \& as } x \text{ goes to -1, } x \text{ goes to -1)}$$

$$= \frac{7\left(\lim_{x\to -1} x\right)^2-5(-1)+7}{-9} \quad \text{(as } x \text{ goes to -1, } x \text{ goes to -1 \& limit of a power of a function is the power of the limit of the function)}$$

$$= \frac{7(-1)^2+5+7}{-9} = -\frac{19}{9} \quad \text{(as } x \text{ goes to -1, } x \text{ goes to -1)}$$

Appendix E - Common Assignment Problems

14. Determine the value of the limit $\lim_{x\to 1}\frac{x-1}{x^2+7x-8}$ = **1/9**

$$\lim_{x\to 1}\frac{x-1}{x^2+7x-8}=\frac{\lim_{x\to 1}x-1}{\lim_{x\to 1}x^2+7x-8}=\frac{\lim_{x\to 1}x-\lim_{x\to 1}1}{\lim_{x\to 1}x^2+\lim_{x\to 1}7x-\lim_{x\to 1}8}=\frac{1-1}{\left(\lim_{x\to 1}x\right)^2+7\lim_{x\to 1}x-8}=\frac{0}{(1)^2+7(1)-8}$$

$$=\frac{0}{1+7-8}=\text{"}\frac{0}{0}\text{"}$$

(Although this attempt has failed, it tells us that both numerator and denominator contain the factor $(x-1)$. This is obvious in the numerator, but we must factor the denominator to see the factor. Once we cancel the common factor we will try again. Note that since we already know one factor, determining the other factor should be relatively easy.)

$$=\lim_{x\to 1}\frac{(x-1)}{(x-1)(x+8)}=\lim_{x\to 1}\frac{1}{(x+8)}=\frac{\lim_{x\to 1}1}{\lim_{x\to 1}(x+8)}=\frac{1}{\lim_{x\to 1}x+\lim_{x\to 1}8}=\frac{1}{1+8}=\frac{1}{9}$$

15. Determine the value of the limit $\lim_{t\to 1}\frac{t^3-t}{t^2-1}$ = **1**

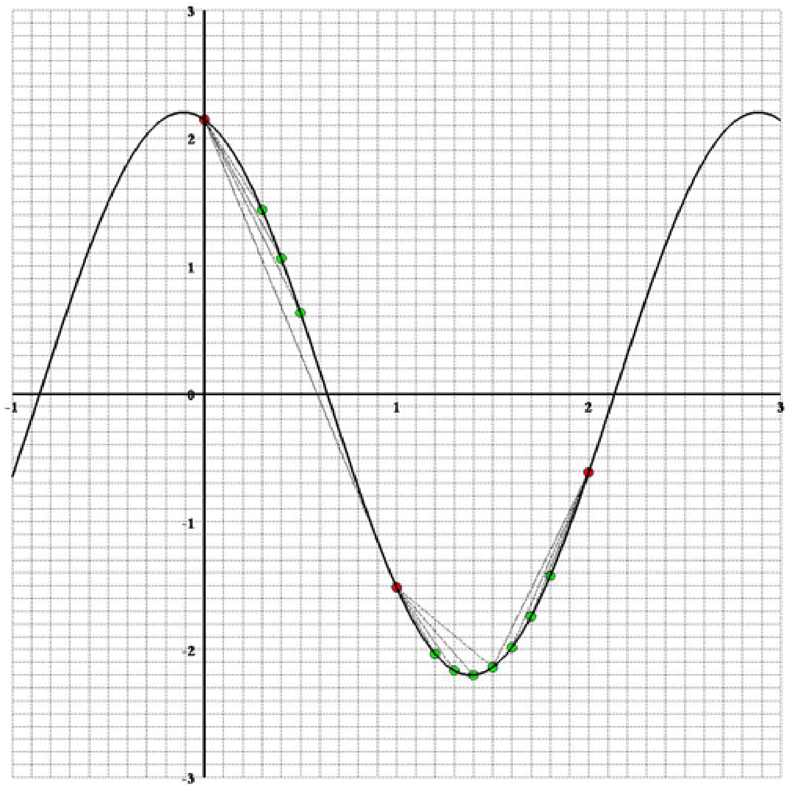
$$\lim_{t\to 1}\frac{t^3-t}{t^2-1}=\frac{\lim_{t\to 1}(t^3-t)}{\lim_{t\to 1}(t^2-1)}=\frac{\lim_{t\to 1}t^3-\lim_{t\to 1}t}{\lim_{t\to 1}t^2-\lim_{t\to 1}1}=\frac{\left(\lim_{t\to 1}t\right)^3-1}{\left(\lim_{t\to 1}t\right)^2-1}=\frac{(1)^3-1}{(1)^2-1}=\text{"}\frac{0}{0}\text{"}$$

$$=\lim_{t\to 1}\frac{t\,(t^2-1)}{(t^2-1)}=\lim_{t\to 1}t=1$$

Appendix E - Common Assignment Problems

Paper Assignment #5 with Solutions

1. The graph below is of the function $f(x) = 2.2 \sin(2.1x + 1.8)$. For each of the following x values draw four secant lines, measure the slopes, and use the sequence of slopes of these lines to estimate the values of the slopes of the tangent lines (also called the derivative):



$f'(0)$: **-2.35258**

The slope of the secant connecting $(0, f(0))$ to $(1, f(1))$ is -3.65555
The slope of the secant connecting $(0, f(0))$ to $(0.5, f(0.5))$ is -3.02000
The slope of the secant connecting $(0, f(0))$ to $(0.4, f(0.4))$ is -2.71164
The slope of the secant connecting $(0, f(0))$ to $(0.3, f(0.3))$ is -2.35258
This last is our best estimate for the derivative at $x = 0$

$f'(1)$: **-2.59855**

The slope of the secant connecting $(1, f(1))$ to $(1.5, f(1.5))$ is -1.25020
The slope of the secant connecting $(1, f(1))$ to $(1.4, f(1.4))$ is -1.71519
The slope of the secant connecting $(1, f(1))$ to $(1.3, f(1.3))$ is -2.16808
The slope of the secant connecting $(1, f(1))$ to $(1.2, f(1.2))$ is -2.59855
This last is our best estimate for the derivative at $x = 1$

$f'(2)$: **4.03959**

The slope of the secant connecting $(2, f(2))$ to $(1.5, f(1.5))$ is 3.04695
The slope of the secant connecting $(2, f(2))$ to $(1.6, f(1.6))$ is 3.42137
The slope of the secant connecting $(2, f(2))$ to $(1.7, f(1.7))$ is 3.75495
The slope of the secant connecting $(2, f(2))$ to $(1.8, f(1.8))$ is 4.03959
This last is our best estimate for the derivative at $x = 2$

2. If $f(x) = 18x + 10$, determine the derivative at $x = -2$.

   Common sense tells us that if the derivative is the slope of the tangent line, and the given function is already linear, then the tangent line and the line which is the graph of the function are one and the same, so their slopes are the same. Thus, the derivative, $f'(x)$, for any x value, must just be the slope of this line, which we can see is 18. However, we should do it anyways using the rules of differentiation, just to check.

$$f'(x) = \frac{df(x)}{dx} = \frac{d(18x+10)}{dx} = \left(\frac{d(18x)}{dx}\right) + \left(\frac{d(10)}{dx}\right) \quad \text{(Sum Rule)}$$

$$= 18\left(\frac{d(x)}{dx}\right) + 0 \quad \text{(Constant Multiple and Constant Rules)}$$

$$= 18(1) = 18 \quad \text{(Identity Rule, Arithmetic)}$$

   Since $f'(x) = 18$, a constant, thus $f'(-2) = 18$ as well.

   $f'(-2) =$ **18**

Appendix E - Common Assignment Problems

3. If $f(x) = 3 + 6x - 5x^2$, determine the derivative at $x = 5$.

$$f'(x) = \frac{df(x)}{dx} = \frac{d(2+6x-5x^2)}{dx} = \left(\frac{d(2)}{dx}\right) + \left(\frac{d(6x)}{dx}\right) - \left(\frac{d(5x^2)}{dx}\right) \quad \text{(Sum and Difference Rules)}$$

$$= 0 + 6\left(\frac{d(x)}{dx}\right) - 5\left(\frac{d(x^2)}{dx}\right) \quad \text{(Constant and Constant Multiple Rules)}$$

$$= 6(1) - 5\left(2x^{2-1}\right) \quad \text{(Identity and Power Rules)}$$

$$= 6\text{-}10x \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

Since $f'(x) = 6 - 10x$, then $f'(5) = 6 - 10(5) = 6 - 50 = -44$

$f'(5) =$ **-44**

4. If $f(x) = 21$, determine the derivative at $x = -11$.

Common sense tells us that if the derivative is the slope of the tangent line, and the given function is a constant (so its graph is a horizontal line), then the tangent line and the line which is the graph of the function are one and the same, so their slopes are the same, 0. Thus, the derivative, $f'(x)$, for any x value, must just be 0. However, we should do it anyways using the rules of differentiation, just to check.

$$f'(x) = \frac{df(x)}{dx} = \frac{d(21)}{dx} = 0 \quad \text{(Constant Rule)}$$

Since $f'(x) = 0$, a constant, thus $f'(-11) = 0$ as well.

$f'(-11) =$ **0**

5. If $f(x) = \frac{2}{x^2}$, determine the derivative at $x = 5$.

So far we did not have any rules relating to functions written as quotients. However, we notice that a little algebra would allow us to rewrite this function so that the rules we already know would apply.

This is a general lesson of Calculus (I, II, and III). We use algebra/functions to rewrite problems before we begin, and as we are doing, Calculus, and the goal is always to make the Calculus either easier or maybe just possible (where before it seemed impossible). This is a critical lesson to learn. All the time in High School when your teacher set you examples of "simplifying" or "expanding" or "factoring", it was always difficult (at least for me when I was a student) to understand exactly what the teacher wanted as the answer. That is, when was the function "simple enough"? In Calculus, it is not the teacher who will say simplify, it has to be you - and it is simple enough, when you can see how to do the Calculus steps that you need to carry out, or when you can see that the Calculus steps are now as simple as possible. You now have to make the decision.

$f(x) = \frac{2}{x^2} \Leftrightarrow f(x) = 2\left(x^{-2}\right)$ Now we use the rules of differentiation:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(2\left(x^{-2}\right)\right)}{dx} = 2\frac{d\left(x^{-2}\right)}{dx} \quad \text{(Constant Multiple Rule)}$$

$$= 2\left(-2x^{-2-1}\right) \quad \text{(Power Rule)}$$

$$= -4x^{-3} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

Since $f'(x) = -4x^{-3}$, so $f'(5) = -4(5)^{-3} = -4/125$

$f'(5) =$ **-4/125**

6. If $f(x) = (6x^2 - 7)(6x + 6)$, determine the derivative, $f'(x)$.

Once again we do not as yet have a rule for dealing with functions that are products like this one. However, if we use algebra and "expand" or "multiply out" this expression, it will be a sum and difference of constant multiples of powers of x, all of which we can deal with using rules that we have learned. Hence, we decide to multiply out:

$$f(x) = (6x^2 - 7)(6x + 6) = 6x^2(6x + 6) - 7(6x + 6) = 36x^3 + 36x^2 - 42x - 42$$

Now we use the rules of differentiation to compute the derivative, $f'(x)$

Appendix E - Common Assignment Problems

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(36x^3 + 36x^2 - 42x - 42\right)}{dx}$$
$$= \frac{d\left(36x^3\right)}{dx} + \frac{d\left(36x^2\right)}{dx} - \frac{d\left(42x\right)}{dx} - \frac{d\left(42\right)}{dx} \quad \text{(Sum \& Difference Rules)}$$
$$= 36\frac{d\left(x^3\right)}{dx} + 36\frac{d\left(x^2\right)}{dx} - 42\frac{d\left(x\right)}{dx} - 0 \quad \text{(Constant Multiple \& Constant Rules)}$$
$$= 36\left(3x^{3-1}\right) + 36\left(2x^{2-1}\right) - 42\left(1\right) \quad \text{(Power \& Identity Rules)}$$
$$= 108x^2 + 72x - 42 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$f'(x)$ = $\mathbf{108x^2 + 72x - 42}$

7. If $f(t) = \frac{\sqrt{3}}{t^3}$, determine the derivative, $f'(t)$.

Once again we do not as yet have a rule for dealing with functions that are products like this one. However, if we use algebra to rewrite it then we will have a function which we can differentiate:

$$f(t) = \frac{\sqrt{3}}{t^3} \Leftrightarrow f(t) = \sqrt{3}\left(t^{-3}\right)$$

Now, using the rules of differentiation, we obtain:

$$f'(t) = \frac{df(t)}{dt} = \frac{d\left(\sqrt{3}\left(t^{-3}\right)\right)}{dt} = \sqrt{3}\frac{d\left(t^{-3}\right)}{dt} \quad \text{(Constant Multiple Rule)}$$
$$= \sqrt{3}\left(-3t^{-3-1}\right) \quad \text{(Power Rule)}$$
$$= -3\sqrt{3}t^{-4} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$f'(t)$ = $\mathbf{-3\sqrt{3}t^{-4}}$

8. The slope of the tangent line to the parabola $y = 2x^2 - 6x + 3$ at the point where $x = -3$ is: **-18**
The equation of this tangent line can be written in the form $y = mx + b$ where m is: **-18**
and where b is: **-15**

$$y' = \frac{dy}{dx} = \frac{d\left(2x^2 - 6x + 3\right)}{dx} = \frac{d\left(2x^2\right)}{dx} - \frac{d\left(6x\right)}{dx} + \frac{d\left(3\right)}{dx} \quad \text{(Sum \& Difference Rules)}$$
$$= 2\frac{d\left(x^2\right)}{dx} - 6\frac{d\left(x\right)}{dx} + 0 \quad \text{(Constant Multiple \& Constant Rules)}$$
$$= 2\left(2x^{2-1}\right) - 6\left(1\right) \quad \text{(Power \& Identity Rules)}$$
$$= 4x - 6 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

Thus, $y'(-3) = \left.\frac{dy}{dx}\right|_{x=-3} = 4(-3) - 6 = -18$

To compute the equation of the tangent line to the parabola at $x = -3$, we must first compute the value of y at $x = -3$: $y(-3) = 2(-3)^2 - 6(-3) + 3 = 2(9) + 18 + 3 = 39$. Now the equation of the tangent line:

$$\frac{y - 39}{x - (-3)} = -18 \Leftrightarrow y - 39 = -18(x + 3) \Leftrightarrow y = -18x - 15$$

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9. If $f(x) = 3x + 3\sqrt{x}$, determine $f'(5)$.
   Once again we do not as yet have a rule for dealing with functions that are products like this one. However, if we use algebra to rewrite it then we will have a function which we can differentiate: $f(x) = 3x + 3\sqrt{x} \Leftrightarrow f(x) = 3x + 3x^{1/2}$
   Now, using the rules of differentiation, we obtain:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(3x + 3x^{1/2}\right)}{dx} = \frac{d\left(3x\right)}{dx} + \frac{d\left(3x^{1/2}\right)}{dx} \quad \textbf{(Sum Rule)}$$

$$= 3\frac{d\left(x\right)}{dx} + 3\frac{d\left(x^{1/2}\right)}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$= 3\left(1\right) + 3\left(\tfrac{1}{2}x^{1/2-1}\right) \quad \textbf{(Identity \& Power Rule)}$$

$$= 3 + \frac{3}{2}x^{-1/2} \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

   Now, substituting in $x = 5$: $f'(5) = \left.\frac{df(x)}{dx}\right|_{x=5} = 3 + \frac{3}{2}(5)^{-1/2} \doteq 3.67082$

   $f'(5) \doteq$ **3.67082**

   Now, for the equation of the tangent line: $\frac{y - 21.70820}{x - 5} = 3.67082 \Leftrightarrow y = 3.60782x + 3.354098$

Use the derivative determined above to determine an equation for the tangent line to the curve $f(x) = 3x + 3\sqrt{x}$ at the point (5, 21.70820). The equation of this tangent line can be written in the form $y = mx + b$ where m is **3.67082** and where b is **3.354098**.

10. The slope of the tangent line to the curve $f(x) = \frac{2}{x}$ at the point (6,0.3333) is: **-0.055556**

Once again we do not as yet have a rule for dealing with functions that are products like this one. However, if we use algebra to rewrite it then we will have a function which we can differentiate:

$$f(x) = \frac{2}{x} \Leftrightarrow f(x) = 2\left(x^{-1}\right)$$

Now we can differentiate:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(2\left(x^{-1}\right)\right)}{dx} = 2\frac{d\left(x^{-1}\right)}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$= 2\left((-1)x^{-1-1}\right) \quad \textbf{(Power Rule)}$$

$$= -2x^{-2} \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

Thus, $f'(6) = \left.\frac{df(x)}{dx}\right|_{x=6} = -2(6)^{-2} = \frac{-2}{36} = \frac{-1}{18} \doteq -0.055556$ and the equation of the tangent line is:

$\frac{y - 0.333333}{x - 6} = -0.055556 \Leftrightarrow y = -0.055556x - 0.666666$. Thus, the equation of this tangent line can be written in the form $y = mx + b$ where m is **-0.055556** and b is **-0.666666**.

11. It is an icy winter day. You are driving your car around a double curve in the road. If the position of your car was plotted on an xy-grid, then the formula for position on the curved road would be: $y = 4x^3$. Suddenly, when you are at the point (1,4), your car begins to skid off the road, instead following a line tangent to the curve at that point.

To know the equation of a tangent line, of any line for that matter, one needs two pieces of information about the line. In general this information could be the coordinates of two points (typical for a secant line), the coordinates of one point and the slope of the line (typical for tangent lines), the slope and y-intercept of the line (typical in high school, but not often available in calculus), etc. The usual information for a tangent line, the coordinates of one point and the slope of the line, are obtained in the following fashion. One is given the formula for a function, and the x-value of the point of tangency. By substituting that x-value into the formula one obtains the

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y-coordinate of that point of tangency. To obtain the slope, one first computes the formula for the derivative function (using the rules of differentiation). While the derivative is just another name for the slope of the tangent line or slope of the function, this derivative function is a function, not a numerical value of slope. The numerical value of the slope at the point of tangency is computed by substituting in the x-value (x-coordinate) of the point of tangency into the derivative function. Enough abstraction - we should see this process in action.

Given $f(x) = 4x^3$, and $x = 1$, we see that $f(1) = 4(1)^3 = 4$. Thus, the point of tangency is $(1, f(1)) = (1,4)$.

(Note 1: The high school notation of y for functions is problematic, since it is difficult to distinguish between y, the function, and values of that function, and between different values of the function. All are represented by the same letter y. It is for this reason that in Calculus we principally use function notation, $f(x)$.

Note 2: In this problem we were actually given the point (1,4), but in most problems we are only given the x-coordinate. Even if given both coordinates, it is not a bad idea to check that the y-coordinate is correct.)

Now we compute $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{df(x)}{dx} = \frac{d\left(4x^3\right)}{dx} = 4\frac{d\left(x^3\right)}{dx} \quad \text{(Constant Multiple Rule)} \\ &= 4\left(3x^{3-1}\right) \quad \text{(Power Rule)} \\ &= 12x^2 \quad \text{(Arithmetic/Algebra/Functions Cleanup)} \end{aligned}$$

Since $f'(x) = 12x^2$, so $f'(1) = 12(1)^2 = 12$. Thus the slope of the tangent line in contact with f at (1,4) is 12. Now we use the point-slope formula for the equation of the straight line, and substitute in the information given and computed.

$$\frac{y - y_0}{x - x_0} = m \Leftrightarrow \frac{y-4}{x-1} = 12 \Leftrightarrow y - 4 = 12(x-1) \Leftrightarrow y = 12x - 12 + 4 \Leftrightarrow y = 12x - 8$$

What is the equation of that tangent line: $y =$ **12** $\times\, x +$ **(-8)**.

12. You have been following the price of a particular stock over three months. You plotted P (price, \$) versus t (time, months) and noticed that the curve $P(t) = 50 + 50t - 5t^2$ fits your current data almost exactly. You decide to make several calculations using this curve as a model for the stock's behaviour before investing in the stock.

What is the instantaneous rate of change of the price of the stock ($P'(t)$): $\mathbf{P'(t) = 50 - 10t}$

$$\begin{aligned} P'(t) &= \frac{dP(t)}{dt} = \frac{d\left(50 + 50t - 5t^2\right)}{dt} = \frac{d(50)}{dt} + \frac{d(50t)}{dt} - \frac{d\left(5t^2\right)}{dt} \quad \text{(Sum \& Difference Rules)} \\ &= 0 + 50\frac{d(t)}{dt} - 5\frac{d\left(t^2\right)}{dt} \quad \text{(Constant \& Constant Multiple Rules)} \\ &= 50(1) - 5\left(2t^{2-1}\right) \quad \text{(Identity \& Power Rules)} \\ &= 50 - 10t \quad \text{(Arithmetic/Algebra/Functions Cleanup)} \end{aligned}$$

when $t = 3$ (current marginal rate of change of price or $P'(3)$: **20**

$$P'(3) = 50 - 10(3) = 50 - 30 = 20$$

when $t = 4$ (marginal rate of change of price or $P'(4)$: **10**

$$P'(4) = 50 - 10(4) = 50 - 40 = 10$$

when $t = 5$ (marginal rate of change of price or $P'(5)$: **0**

$$P'(5) = 50 - 10(5) = 50 - 50 = 0$$

If you buy the stock now ($t = 3$), when do you think you should sell: at $t = 5$

Clearly the price of the stock is increasing at $t = 3$ and 4 since the slopes of the tangent lines at these points is positive. However, at $t = 5$ the slope of the tangent line is 0, meaning the tangent line is now horizontal, indicating that the price of the stock has stopped rising, which is a good time to sell (before it starts falling).

How much profit will you have made (per share): **20**

Since we buy the stock at $t = 3$, we must pay $P(3) = 50 + 50(3) - 5(3)^2 = 50 + 150 - 45 = 155$. If we sell the stock at $t = 5$, we are paid $P(5) = 50 + 50(5) - 5(5)^2 = 50 + 250 - 125 = 175$. The profit we make per share is the difference, $P(5) - P(3) = 175 - 155 = 20$. Thus, we make \$20 per share.

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13. Sony & BMG Music Entertainment charges different prices (p, \$) to retailers for their new CD releases based on the number of items (x) ordered. Your boss at Sony is tired of using tables to do this calculation so he shows you a function that he has worked out but he asks you to iron out a few bugs. In particular, he wants that when a customer shifts from ordering 999 CD's to 1000, or from 9999 CD's to 10000, the total cost, $C(x) = xp(x)$, should not change abruptly. You realize that his requirement is similar to the notion that the function for total cost be continuous at $x = 1000$ and $x = 10000$. The function that he gives you is:

$$p(x)=\begin{cases} 8-ax & , \quad 0<x<1000 \\ 6-bx & , \quad 1000\le x<10{,}000 \\ 5 & , \quad 10{,}000\le x \end{cases}$$

What values should a and b have so that $C(x)$ is continuous as required?

We begin by writing the function for $C(x)$: $C(x)=xp(x)=\begin{cases} 8x-ax^2 & , \quad 0<x<1000 \\ 6x-bx^2 & , \quad 1000\le x<10{,}000 \\ 5x & , \quad 10{,}000\le x \end{cases}$

We note that $C(x)$ is a piecewise defined function, and each of the pieces is a polynomial. In fact, two pieces are quadratic functions and the third piece is a linear function. Now polynomials are continuous everywhere. Thus, the only values of x at which $C(x)$ could be discontinuous are the "seams", where the pieces meet. Here, that is at $x = 1000$ and $x = 10{,}000$.

Continuity at $x = 1000$:

1) Does $\lim\limits_{x\to 1000} C(x)$ exist (a finite number)?

Unfortunately, unless we know on which side of 1000 the value of x lies, we do not know which formula to use in this limit. To solve this we break this limit into two half-limits or one-sided limits.

$$\begin{aligned}\lim_{x\to 1000^-} C(x) &= \lim_{x\to 1000^-}\left(8x-ax^2\right)\\ &= 8(1000)-a(1000)^2 \ \ (8x-ax^2 \textbf{ is a polynomial, hence continuous})\\ &= 8{,}000-1{,}000{,}000a\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1000^+} C(x) &= \lim_{x\to 1000^+}\left(6x-bx^2\right)\\ &= 6(1000)-b(1000)^2 \ \ (6x-bx^2 \textbf{ is a polynomial, hence continuous})\\ &= 6{,}000-1{,}000{,}000b\end{aligned}$$

According to a theorem concerning limits, the whole limit exists if and only if the two half-limits exist and are equal. Thus, $\lim\limits_{x\to 1000} C(x)$ exists if and only if $\lim\limits_{x\to 1000^-} C(x)$ and $\lim\limits_{x\to 1000^+} C(x)$ exist and are equal. We see that this will only be true if 8,000 - 1,000,000a = 6,000 - 1,000,000b ⇔ 4 - 500a = 3 - 500b (EQ #1)

We will need a second equation if we are to solve for unique answers for a and b.

Continuity at $x = 10{,}000$:

1) Does $\lim\limits_{x\to 10000} C(x)$ exist (a finite number)?

Just as above, unless we know on which side of 10,000 the value of x lies, we do not know which formula to use in this limit. To solve this we break this limit into two half-limits or one-sided limits.

$$\begin{aligned}\lim_{x\to 10000^-} C(x) &= \lim_{x\to 10000^-}\left(6x-bx^2\right)\\ &= 6(10000)-b(10000)^2 \ \ (6x-bx^2 \textbf{ is a polynomial, hence continuous})\\ &= 60{,}000-10{,}000{,}000{,}000b\end{aligned}$$

$$\lim_{x\to 10000^+} C(x) = \lim_{x\to 10000^+}\left(5x\right) = 5(10000) \ \ (5x \textbf{ is a polynomial, hence continuous}) = 50{,}000$$

As above, $\lim\limits_{x\to 10000} C(x)$ exists if and only if $\lim\limits_{x\to 10000^-} C(x)$ and $\lim\limits_{x\to 10000^+} C(x)$ exist and are equal. We see that this will only be true if

60,000 - 100,000,000b = 50,000 ⇔ 10,000 = 100,000,000b ⇔ b = 1/10,000 = 0.0001 (EQ #2)

Substituting this second result, EQ #2, into the first result, EQ #1:

4 - 500a = 3 - 500(1/10,000) ⇔ -500a = -1 - 5/100 ⇔ a = 0.0021

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We should complete our check that the conditions for continuity are met at each of x = 1000 and x = 10,000.

Continuity at x = 1,000:
1) Does $\lim_{x\to 1000} C(x)$ exist (a finite number)?

We now know that $\lim_{x\to 1000} C(x)$ = 8,000 - 1,000,000a = 6,000 - 1,000,000b = 5,900

2) Does C(1,000) exist?
We now know that $C(1{,}000) = 6(1{,}000) - 0.0001(1{,}000)^2 = 6{,}000 - 100 = 5{,}900$
3) Does $\lim_{x\to 1000} C(x) = C(1{,}000)$?

We see from the above computation that the limit and the value of the function are the same.
Continuity at x = 10,000:
1) Does $\lim_{x\to 10000} C(x)$ exist (a finite number)?

We now know that $\lim_{x\to 10000} C(x)$ = 60,000 - 100,000,000b = 50,000

2) Does C(10,000) exist?
We now know that C(10,000) = 5(10,000) = 50,000
3) Does $\lim_{x\to 10000} C(x) = C(10{,}000)$?

We see from the above computation that the limit and the value of the function are the same.
Thus, as long as a = 0.0021 and b = 0.0001, the function C is continuous everywhere.

a: **0.0021** *b*: **0.0001**

14. Given $f(x)=\begin{cases}\dfrac{6x^3+31x^2+12x+35}{x+5}, & 0<x<-5\\ -5x^2-1x+a, & -5\le x\end{cases}$, what value should a have in order that f is continuous at x = -5?

a: **272**

This problem is similar to that in 13, but there are differences. The most obvious difference is that there is only one unknown constant, a. This will mean that we need only generate one equation in a to be able to solve for a. The second difference is that unlike 13, where all of the functions involved were polynomials, one of the two functions here is a rational function. Unlike polynomial functions, rational functions are not continuous everywhere, they are just continuous where the denominator is not zero. However, the rational function here, $\dfrac{6x^3+31x^2+12x+35}{x+5}$, would have a discontinuity at x = -5 (just solve x + 5 = 0), and x = -5 is already the "seam" where the two pieces of this function "meet". Thus, the only point that we need to verify continuity at is x = -5.
Continuity at x = -5
1) Does $\lim_{x\to -5} f(x)$ exist (finite number)?

Since x = -5 is the "seam" we cannot compute this limit directly, not knowing which piece of the function to use unless we know on which side of -5 the value of x lies. Thus, as in 13 above, we split this limit into two half-limits.

$$\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} \frac{6x^3+31x^2+12x+35}{x+5} =$$
$$= \text{"}\frac{6(-5)^3+31(-5)^2+12(-5)+35}{(-5)+5}\text{"} = \text{"}\frac{-750+775-60+35}{(-5)+5}\text{"} = \text{"}\frac{0}{0}\text{"}$$

$$\begin{array}{r} 6x^2 + x + 7 \\ (x+5)\overline{)\,6x^3+31x^2+12x+35} \\ \underline{6x^3+30x^2} \qquad\qquad \\ x^2+12x \qquad \\ \underline{x^2+5x} \qquad \\ 7x+35 \\ \underline{7x+35} \\ 0 \end{array}$$

While this looks like a dead end, this work actually provides direction for our next step towards a solution. A polynomial function (such as the numerator and denominator here) can only equal 0 when x is replaced by -5 if (x + 5) is a factor. We can clearly see the factor (x + 5) in the denominator, but we now know that the numerator must also contain such a factor. Thus, our next step is to factor the numerator, then cancel the common factor of (x + 5) and then attempt again to evaluate the limit. We note that the numerator polynomial is a cubic, and there does not seem to be a simple technique for factoring it (grouping terms does not look promising). However, we know that (x + 5) is a factor. Thus, we resort to long division. This is just like long division with numbers.

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That is, we now know that $6x^3 + 31x^2 + 12x + 35 = (x + 5)(6x^2 + x + 7)$ and using this we attempt again to compute the it that we started.

$$\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} \frac{6x^3+31x^2+12x+35}{x+5} = \lim_{x\to -5^-} \frac{\cancel{(x+5)}\left(6x^2+x+7\right)}{\cancel{(x+5)}}$$

$$= \lim_{x\to -5^-}\left(6x^2+x+7\right) = 6(-5)^2+(-5)+7 = 150-5+7 = 152$$

Now we must compute the other half-limit.

$$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+}\left(-5x^2-1x+a\right) = -5(-5)^2-(-5)+a = -125+5+a = a-120$$

For the whole limit to exist, the two half-limits must be equal. Thus, we need: $a - 120 = 152 \Leftrightarrow a = 272$

2) Does $f(-5)$ exist?

$f(-5) = -5(-5)^2 - (-5) + a = -125 + 5 + a = a - 120 = 272 - 120 = 152$

3) Does $\lim_{x\to -5} f(x) = f(-5)$?

Our work above shows that the limit and the value of the function are the same.

15. Determine a and b such that the function $f(x) = \begin{cases} x^2-4x+7 & \text{if} \quad x\le 3 \\ ax+b & \text{if} \quad 3<x \end{cases}$ is differentiable everywhere.

a = __________ b = __________

For a function to be differentiable it must also be continuous. Once again our function is piecewise polynomial, *i.e.*, each piece of the function is a polynomial. Now polynomial functions are continuous and differentiable everywhere. Thus, the only value of x where f could fail to be continuous or differentiable is at the "seam", *i.e.*, $x = 3$. We begin by checking for continuity at $x = 3$.

1) Does $\lim_{x\to 3} f(x)$ exist (a finite number)?

Unless we know which side of 3 the values of x lie we cannot determine which formula to use for f. Thus, we are forced to split this limit into two half-limits.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\left(x^2-4x+7\right) = (3)^2-4(3)+7 = 9-12+7 = 4$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(ax+b) = a(3)+b = 3a+b$$

Thus, $\lim_{x\to 3} f(x)$ exists if and only if $3a + b = 4$ (EQ #1). This is a single equation in two unknown constants, a and b. We will need a second equation to be able to solve for unique values for a and b.

2) Does $f(3)$ exist?

$f(3) = (3)^2 - 4(3) + 7 = 4$

3) Does $\lim_{x\to 3} f(x) = f(3)$?

We can see from our work above, that as long as $3a + b = 4$, then the limit equals the value of the function at the point.

Now we must check for differentiability (existence of the derivative or slope of the tangent line) at $x = 3$.

We go back to the definition of derivative, the limit of Newton's Quotient, that is:

$$f'(3) = \left.\frac{df(x)}{dx}\right|_{x=3} = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{f(3+h)-4}{h}$$

We cannot evaluate $f(3 + h)$ unless we know whether $3 + h \le 3 \Leftrightarrow h \le 0$ or $3 < 3 + h \Leftrightarrow 0 < h$, because we do not know which formula to use in the limit. Thus, we must break this limit into two half-limits.

$$f'_-(3) = \left.\frac{df(x)}{dx}\right|_{x=3^-} = \lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-}\frac{((3+h)^2-4(3+h)+7)-4}{h}$$

$$= \lim_{h\to 0^-}\frac{(9+\overset{2h}{\cancel{6h}}+h^2-12-\cancel{4h}+7)-4}{h} = \lim_{h\to 0^-}\frac{2h+h^2+\cancel{4}-\cancel{4}}{h} = \lim_{h\to 0^-}\frac{2h+h^2}{h}$$

$$= \lim_{h\to 0^-}\frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h\to 0^-}(2+h) = 2+0 = 2$$

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$$f'_+(3) = \left.\frac{df(x)}{dx}\right|_{x=3^+} = \lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+}\frac{(a(3+h)+b)-4}{h} = \lim_{h\to 0^+}\frac{3a+ah+b-4}{h}$$

$$= \lim_{h\to 0^+}\frac{(3a+b-4)+ah}{h} = \lim_{h\to 0^+}\frac{0+ah}{h} = \lim_{h\to 0^+}\frac{a\not{h}}{\not{h}} = \lim_{h\to 0^+} a = a$$

Note that in the second half-limit we had to use the previously established information that $3a + b = 4$ or its equivalent, that $(3a + b - 4) = 0$, or we would not have been able to complete the calculation. Now, for the whole derivative to exist, instead of just these two half-derivatives, we need these two limits to be equal. That is, $a = 2$. Substituting this into $3a + b = 4$ we have $6 + b = 4 \Leftrightarrow b = -2$. Thus, we see that if $a = 2$ and $b = -2$, then f is both continuous and differentiable everywhere.

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Paper Assignment #6 with Solutions

1. Let $f(x) = -2x^3 - 5x + 1$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\,-2x^3 - 5x + 1}{dx} = \frac{d\,-2x^3}{dx} - \frac{d\,5x}{dx} + \frac{d1}{dx} \text{ (Sum \&Difference Rules)}$$

$$= -2\frac{dx^3}{dx} - 5\frac{dx}{dx} + 0 \text{ (Constant Multiple \& Constant Rules)}$$

$$= -2\left(3x^{3-1}\right) - 5(1) \text{ (Power \& Identity Rules)}$$

$$= -6x^2 - 5 \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

The first derivative of $f(x)$: $\frac{df}{dx}(x) = f'(x) = \mathbf{-6x^2 - 5}$

$$f'(x) = \frac{df(x)}{dx} = -6x^2 - 5$$

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d\,-6x^2 - 5}{dx} = \frac{d\,-6x^2}{dx} - \frac{d\,5}{dx} \text{ (Sum \&Difference Rules)}$$

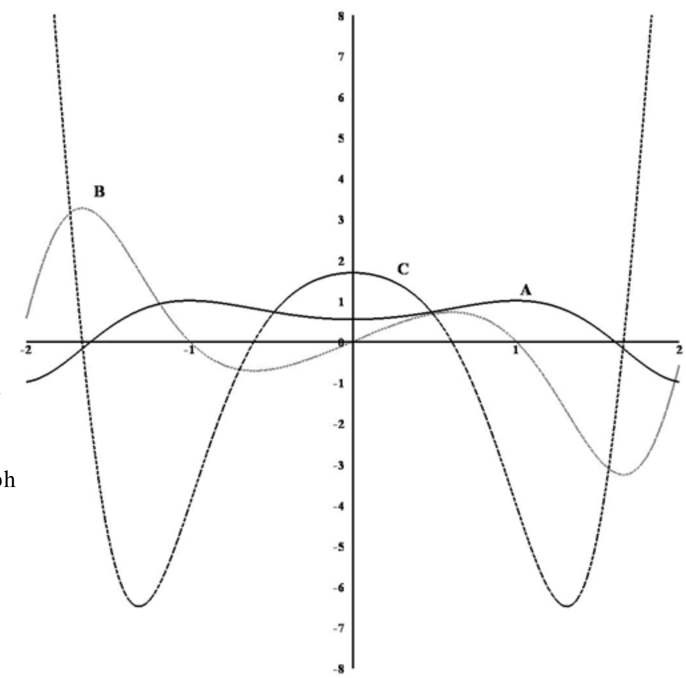
$$= -6\frac{dx^2}{dx} - 0 \text{ (Constant Multiple \& Constant Rules)}$$

$$= -6\left(2x^{2-1}\right) \text{ (Power Rule)}$$

$$= -12x \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

The second derivative of $f(x)$: $\frac{d^2 f}{dx^2}(x) = f''(x) = \mathbf{-12x}$

2. The grid on the right contains graphs of three different functions, $f(x)$, $\frac{df}{dx}(x) = f'(x)$, $\frac{d^2 f}{dx^2}(x) = f''(x)$. Indicate below which graph (A solid black; B dotted grey; C dashed black) represents:



We know that wherever the slope of a tangent line to a graph of f is 0 (*i.e.*, where f has a horizontal tangent line), the derivative of f, f', will have a zero and its graph will therefore have an x-intercept. Similarly, wherever the slope of a tangent line to a graph of f' is 0, the derivative of f', f'', will have a zero and its graph will therefor have an x-intercept. However, wherever a graph of f changes concavity (called having a "point of inflection"), a graph of f'' will have an x-intercept. Looking at the given graphs we note that where graph A has horizontal tangents (around $x = -1, 0, 1$), graph B has x-intercepts. We note also that where graph B has horizontal tangents (around $x = -1.75, -0.6, 0.6, 1.75$), graph C has x-intercepts. Finally, where graph C has x-intercepts, graph A appears to be changing concavity. Thus, graph A is of f, graph B is of f', and graph C is of f''.

Graph A is the graph of the function $f(x)$

Graph B is the graph of the first derivative, function $\frac{df}{dx}(x) = f'(x)$

Graph C is the graph of the second derivative, function $\frac{d^2 f}{dx^2}(x) = f''(x)$

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3. Let $f(x) = x^4 + 2x^3 + 5x^2 + 3x$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{dx^4 + 2x^3 + 5x^2 + 3x}{dx} = \frac{dx^4}{dx} + \frac{d2x^3}{dx} + \frac{d5x^2}{dx} + \frac{d3x}{dx} \quad \text{(Sum \&Difference Rules)}$$

$$= \left(4x^{4-1}\right) + 2\frac{dx^3}{dx} + 5\frac{dx^2}{dx} + 3\frac{dx}{dx} \quad \text{(Power \& Constant Multiple Rules)}$$

$$= 4x^3 + 2\left(3x^{3-1}\right) + 5\left(2x^{2-1}\right) + 3(1) \quad \text{(Power \& Identity Rules)}$$

$$= 4x^3 + 6x^2 + 10x + 3 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d4x^3 + 6x^2 + 10x}{dx} = \frac{d4x^3}{dx} + \frac{d6x^2}{dx} + \frac{d10x}{dx} \quad \text{(Sum \&Difference Rules)}$$

$$= 4\frac{dx^3}{dx} + 6\frac{dx^2}{dx} + 10\frac{dx}{dx} \quad \text{(Constant Multiple Rule)}$$

$$= 4\left(3x^{3-1}\right) + 6\left(2x^{2-1}\right) + 10(1) \quad \text{(Power \& Identity Rules)}$$

$$= 12x^2 + 12x + 10 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'(1) = \left.\frac{df(x)}{dx}\right|_{x=1} = 4(1)^3 + 6(1)^2 + 10(1) + 3 = 23$$

$$f''(1) = \left.\frac{d^2 f(x)}{dx^2}\right|_{x=1} = 12(1)^2 + 12(1) + 10 = 34$$

$\frac{df}{dx}(x) = f'(x) = \mathbf{4x^3 + 6x^2 + 10x + 3}$ and $\left.\frac{df(x)}{dx}\right|_{x=1} = f'(1) = \mathbf{23}$

$\frac{d^2 f}{dx^2}(x) = f''(x) = \mathbf{12x^2 + 12x + 10}$ and $\left.\frac{d^2 f(x)}{dx^2}\right|_{x=1} = f''(1) = \mathbf{34}$

Appendix E - Common Assignment Problems

4. Let $f(x)=5+\frac{5}{x}+\frac{5}{x^2}$. Use the **rules of differentiation** to compute:

$$f'(x)=\frac{df(x)}{dx}=\frac{d\left(5+\frac{5}{x}+\frac{5}{x^2}\right)}{dx}=\frac{d\left(5+5x^{-1}+5x^{-2}\right)}{dx}\quad \textbf{(Algebra to rewrite function)}$$

$$=\frac{d5}{dx}+\frac{d5x^{-1}}{dx}+\frac{d5x^{-2}}{dx}\quad \textbf{(Sum \& Difference Rules)}$$

$$=0+5\frac{dx^{-1}}{dx}+5\frac{dx^{-2}}{dx}\quad \textbf{(Constant \& Constant Multiple Rules)}$$

$$=5\left((-1)x^{-1-1}\right)+5\left(-2x^{-2-1}\right)\quad \textbf{(Power Rule)}$$

$$=-5x^{-2}-10x^{-3}\quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f''(x)=\frac{d^2f(x)}{dx^2}=\frac{d-5x^{-2}-10x^{-3}}{dx}=\frac{d-5x^{-2}}{dx}-\frac{d10x^{-3}}{dx}\quad \textbf{(Sum \&Difference Rules)}$$

$$=-5\frac{dx^{-2}}{dx}-10\frac{dx^{-3}}{dx}\quad \textbf{(Constant Multiple Rule)}$$

$$=-5\left(-2x^{-2-1}\right)-10\left(-3x^{-3-1}\right)\quad \textbf{(Power Rules)}$$

$$=10x^{-3}+30x^{-4}\quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'(4)=\left.\frac{df(x)}{dx}\right|_{x=4}=-5(4)^{-2}-10(4)^{-3}=-\left(\frac{5}{4^2}+\frac{10}{4^3}\right)=-\left(\frac{5}{4^2}\times\frac{4}{4}+\frac{10}{4^3}\right)=-\left(\frac{30}{64}\right)=-\left(\frac{15}{32}\right)$$

$$f''(4)=\left.\frac{d^2f(x)}{dx^2}\right|_{x=4}=10(4)^{-3}+30(4)^{-4}=\left(\frac{10}{4^3}+\frac{30}{4^4}\right)=\left(\frac{10}{4^3}\times\frac{4}{4}+\frac{30}{4^4}\right)=\left(\frac{70}{4^4}\right)=\left(\frac{70}{256}\right)=\left(\frac{35}{128}\right)$$

$\frac{df}{dx}(x)=f'(x)=-5x^{-2}-10x^{-3}$ and $\left.\frac{df(x)}{dx}\right|_{x=4}=f'(4)=-\left(\frac{15}{32}\right)$

$\frac{d^2f}{dx^2}(x)=f''(x)=10x^{-3}+30x^{-4}$ and $\left.\frac{d^2f(x)}{dx^2}\right|_{x=4}=f''(4)=\left(\frac{35}{128}\right)$

Appendix E - Common Assignment Problems

5. Let $g(t) = 2t^4 - 3t^2 - 2$. Use the **rules of differentiation** to compute:

$$g'(t) = \frac{dg(t)}{dt} = \frac{d\,2t^4 - 3t^2 - 2}{dt} = \frac{d\,2t^4}{dt} - \frac{d\,3t^2}{dt} - \frac{d\,2}{dt} \text{ (Sum \& Difference Rules)}$$

$$= 2\frac{dt^4}{dt} - 3\frac{dt^2}{dt} - 0 \text{ (Constant Multiple \& Constant Rules)}$$

$$= 2\left(4t^{4-1}\right) - 3\left(2t^{2-1}\right) \text{ (Power Rule)}$$

$$= 8t^3 - 6t \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

$$g''(t) = \frac{d^2g(t)}{dt^2} = \frac{d\,8t^3 - 6t}{dt} = \frac{d\,8t^3}{dt} - \frac{d\,6t}{dt} \text{ (Sum \& Difference Rules)}$$

$$= 8\frac{dt^3}{dt} - 6\frac{dt}{dt} \text{ (Constant Multiple Rule)}$$

$$= 8\left(3t^{3-1}\right) - 6\left(1\right) \text{ (Power \& Identity Rules)}$$

$$= 24t^2 - 6 \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

$$g'''(t) = \frac{d^3g(t)}{dt^3} = \frac{d\,24t^2 - 6}{dt} = \frac{d\,24t^2}{dt} - \frac{d\,6}{dt} \text{ (Sum \& Difference Rules)}$$

$$= 24\frac{dt^2}{dt} - 0 \text{ (Constant Multiple \& Constant Rules)}$$

$$= 24\left(2t^{2-1}\right) \text{ (Power Rule)}$$

$$= 48t \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

$$g^{(4)}(t) = \frac{d^4g(t)}{dt^4} = \frac{d\,48t}{dt} = 48\frac{dt}{dt} \text{ (Constant Multiple Rule)}$$

$$= 48\left(1\right) \text{ (Identity Rule)}$$

$$= 48 \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

$$g^{(5)}(t) = \frac{d^5g(t)}{dt^5} = \frac{d\,48}{dt} = 0 \text{ (Constant Rule)}$$

$$g(0) = 2(0)^4 - 3(0)^2 - 2 = -2$$

$$g'(0) = 8(0)^3 - 6(0) = 0$$

$$g''(0) = 24(0)^2 - 6 = -6$$

$$g'''(0) = 48(0) = 0$$

$$g^{(4)}(0) = 48$$

$$g^{(5)}(0) = 0$$

$g(0) =$ **2** $g'(0) =$ **0** $g''(0) =$ **-6** $g'''(0) =$ **0** $g^{(4)}(0) =$ **48** $g^{(5)}(0) =$ **0**

Appendix E - Common Assignment Problems

6. Use linear approximation, *i.e.*, a tangent line, to approximate $\sqrt[3]{27.4}$ as follows:

   Let $f(x) = \sqrt[3]{x}$. The equation of the tangent line to $f(x)$ at $x = 27$ can be written in the form

   $g(x) = mx + b$ where m is: **1/27** and where b is: **2**

   Using this tangent line, your approximation for $\sqrt[3]{27.4} \doteq g(27.4) \doteq$ **3.014815**

$$f'(x) = \frac{df(x)}{dx} = \frac{d\sqrt[3]{x}}{dx} = \frac{dx^{1/3}}{dx} \quad \text{(Algebra to rewrite function)}$$
$$= \left(\frac{1}{3}\right)x^{1/3-1} \quad \text{(Power Rule)}$$
$$= \left(\frac{1}{3}\right)x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\left(\sqrt[3]{x}\right)^2}$$
$$f(27) = \sqrt[3]{27} = 3;\ f'(27) = \frac{1}{3\left(\sqrt[3]{27}\right)^2} = \frac{1}{3(3)^2} = \frac{1}{27}$$
$$\frac{y-3}{x-27} = \frac{1}{27} \Leftrightarrow y - 3 = \frac{1}{27}(x-27) \Leftrightarrow y = \frac{1}{27}x + 2$$
$$g(x) = \frac{1}{27}x + 2 \Rightarrow g(27.4) = \frac{27.4}{27} + 2 = 3 + \frac{4}{270} \doteq 3.01485$$

Note: Using a spreadsheet to compute $\sqrt[3]{27.4}$ one obtains an answer of 3.014742. Thus, our estimate was off by 0.000073, or 0.002407% overestimated. We could have predicted that our approximation was an overestimate because a graph of *f* is concave down everywhere, so that tangent lines to the curve lie above the curve. Hence heights on tangent lines are higher than heights on the curve.

7. You have \$10,000 to invest in an annually compounded GIC, locked in for 7 years. The bank offers you an annual interest rate of 3.5%. You forgot your calculator at home, so you decide to do a quick approximation of how much money you will have at the end of the 7 years by doing a linear approximation. You say that the amount of money is $V(x) = 10000 \times x^7$, where x is 1.035. The equation of the tangent line to $V(x)$ at $x = 1$ can be written in the form

   $g(x) = mx + b$ where m is: **70000** and where b is: **-60000**

   Using this tangent line, your approximation for the value of the GIC after seven years,

   $\doteq g(1.035) =$ **12450**

   The bank's financial advisor notices you doing your calculation by hand so he offers you his calculator. Using his calculator you compute the exact value of the GIC after seven years as: **12722.79**

$$V'(x) = \frac{dV(x)}{dx} = \frac{d10000x^7}{dx} = 10000\frac{dx^7}{dx} \quad \text{(Constant Multiple Rule)}$$
$$= 10000\left(7x^{7-1}\right) \quad \text{(Power Rule)}$$
$$= 70000x^6 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$V(1) = 10000(1)^7 = 10000;\ V'(1) = 70000(1)^6 = 70000$$
$$\frac{y-10000}{x-1} = 70000 \Leftrightarrow y - 10000 = 70000(x-1) \Leftrightarrow y = g(x) = 70000x - 60000$$
$$g(1.035) = 70000(1.035) - 60000 = 12450$$
$$V(1.035) = 10000(1.035)^7 \doteq 12722.79$$

Note: Using a spreadsheet we can see that our estimate was about 272.79, or 2.14% underestimated. We could have predicted that our approximation would be an underestimate because a graph of *V* is concave up everywhere, so that tangent lines to the curve lie below the curve. Hence heights on tangent lines are lower than heights on the curve.

Appendix E - Common Assignment Problems

8. Your boss just gave you a \$5,000 bonus. You decide to invest in a bond that compounds annually at 4%. Without your calculator, use a linear approximation to determine how much money you will have at the end of 5 years. If the amount of money is $V(x) = 5000x^5$, where x is 1.04. The equation of the tangent line to $V(x)$ at $x = 1$ can be written in the form $g(x) = mx + b$ where m is: 25000 and where b is: **-20000**
   Using this tangent line, your approximation for the value of the bond after 5 years is
   ≐ g(1.04) = **6000**

$$V'(x) = \frac{dV(x)}{dx} = \frac{d\left(5000x^5\right)}{dx} = 5000\frac{d\left(x^5\right)}{dx} \quad \textbf{(Constant Multiple Rule)}$$
$$= 5000\left(5x^{5-1}\right) \quad \textbf{(Power Rule)}$$
$$= 25000x^4 \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

To compute the tangent line function, $g(x)$, we first need the point of tangency, $(1, f(1)) = (1,5000)$ and the slope of V at that point, namely $V'(1) = 25000$. Then the tangent line to V at $x = 1$ is:

$$\frac{y-5000}{x-1} = 25000 \Leftrightarrow y - 5000 = 25000x - 25000 \Leftrightarrow g(x) = y = 25000x - 20000$$

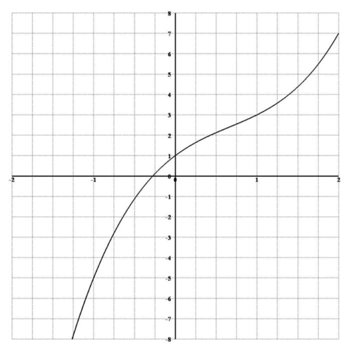
Then our approximation of $V(1.04) \doteq 25000(1.04) - 20000 = 6000$

9. Use linear approximation, *i.e.*, a tangent line, to approximate the value of $\frac{1}{0.203}$ as follows:

   Let $f(x) = \frac{1}{x}$. Determine the equation of the tangent line to $f(x)$ at a "nice" point near 0.203 (you must choose this point, but all calculation should be done without a calculator). Then use this to approximate $\frac{1}{0.203}$ ≐ **4.925**

Given that $f(x) = x^{-1}$, so $f'(x) = (-1)x^{-2}$ (Power Rule). A "nice" value of x near to $x = 0.203$ would be $x = 0.200$. We note that $f(0.200) = (2/10)^{-1} = 5$ and $f'(0.200) = -(2/10)^{-2} = -25$. Thus, the line tangent to f at $x = 0.200$ is:

$\frac{y-5}{x-0.200} = -25 \Leftrightarrow y - 5 = -25x + 5 \Leftrightarrow y = -25x + 10$. Thus, our linear function that we use for approximation of f will be $g(x) = -25x + 10$ and our approximation of $f(0.203) \doteq g(0.203) = -25(0.203) + 10 = 4.925$

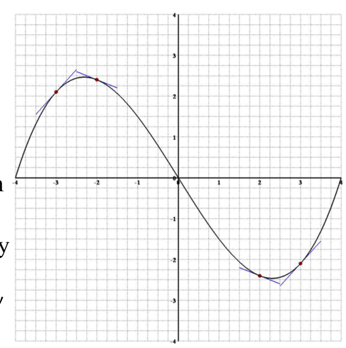


10. There is a function $f(x)$ but all that you know is that $f(1) = 5$ and the graph of the derivative of $f(x)$, $f'(x)$, is as shown on the right.

To construct the tangent line to a curve f at the point where $x = 1$ we need to know two pieces of information: a point on the line and the slope of the line. Since we know that $x = 1$ and $f(1) = 5$, we know that the point of tangency is (1,5). By looking at the graph we can see that $f'(1) = 3$, and this is the slope, m, of the tangent line. Thus, the line tangent to f at $x = 1$ is given by the equation:

$$\frac{y-5}{x-1} = 3 \Leftrightarrow y - 5 = 3x - 3 \Leftrightarrow y = 3x + 2$$

Using function notation, we have $g(x) = 3x + 2$ as the linear function that we will use to approximate $f(x)$, for values of x near to 1. Thus:
$f(0.9) \doteq g(x) = 3(0.9) + 2 = 4.7$



Will this estimate be an over estimate or an under estimate? This will depend upon the concavity of the function f at $x = 1$. That is, if f is concave down at $x = 1$, then the tangent line at $x = 1$ will lie above the curve, and a height on the tangent line will be larger than a height on the graph of f. That is, if f is concave down, then our estimate is an over estimate. Similarly, if f is concave up at $x = 1$, then the tangent line at $x = 1$ will lie below the curve, and a height on the tangent line will be smaller than a height on the graph of f. That is, if f is concave up, then our estimate is an under estimate. The graph on the right illustrates these notions. Of course, we do not have a graph of f so the natural question arises, how do we know if f is concave up or concave down at $x = 1$? We have however seen that the concavity of f is connected to the sign of f'', that is, f is concave up if and only if f'' is positive and f is concave down if and only if f'' is negative. But we don't have a graph of f'' either so how would we know if f'' is positive or negative? What we do have is a graph of f', and slopes of tangent lines to this graph are

Appendix E - Common Assignment Problems

actually values of f''. We can easily observe that at $x = 1$, a tangent line would have a positive slope (because the function f' is seen to be increasing at $x = 1$). Thus, we conclude that $f''(1)$ is positive, so f is concave up at $x = 1$, which in turn means that a tangent to f at $x = 1$ lies underneath a graph of f, so that our linear approximation is thus an under estimate, *i.e.*, too small.
Use linear approximation to estimate $f(0.9)$: **4.7**
Is your answer a little too big or a little too small? **too small**

11. Let $h(t) = 2t^{3.2} - 6t^{-3.2}$. Use the **rules of differentiation** to compute:

$$h'(t) = \frac{dh(t)}{dt} = \frac{d\left(2t^{3.2} - 6t^{-3.2}\right)}{dt} = \frac{d\left(2t^{3.2}\right)}{dt} - \frac{d\left(6t^{-3.2}\right)}{dt} \quad \text{(Difference Rule)}$$
$$= 2\frac{d\left(t^{3.2}\right)}{dt} - 6\frac{d\left(t^{-3.2}\right)}{dt} \quad \text{(Constant Multiple Rule)}$$
$$= 2\left(3.2t^{3.2-1}\right) - 6\left(-3.2t^{-3.2-1}\right) \quad \text{(Power Rule)}$$
$$= 6.4t^{2.2} + 19.2t^{-4.2} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$h''(t) = \frac{d^2h(t)}{dt^2} = \frac{d\left(6.4t^{2.2} + 19.2t^{-4.2}\right)}{dt} = \frac{d\left(6.4t^{2.2}\right)}{dt} + \frac{d\left(19.2t^{-4.2}\right)}{dt} \quad \text{(Sum Rule)}$$
$$= 6.4\frac{d\left(t^{2.2}\right)}{dt} + 19.2\frac{d\left(t^{-4.2}\right)}{dt} \quad \text{(Constant Multiple Rule)}$$
$$= 6.4\left(2.2t^{2.2-1}\right) + 19.2\left(-4.2t^{-4.2-1}\right) \quad \text{(Power Rule)}$$
$$= 14.08t^{1.2} - 80.64t^{-5.2} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$h'(3) = \left.\frac{dh(t)}{dt}\right|_{t=3} = 6.4(3)^{2.2} + 19.2(3)^{-4.2} \doteq 71.94$$
$$h''(3) = \left.\frac{d^2h(t)}{dt^2}\right|_{t=3} = 14.08(3)^{1.2} - 80.64(3)^{-5.2} \doteq 52.35$$

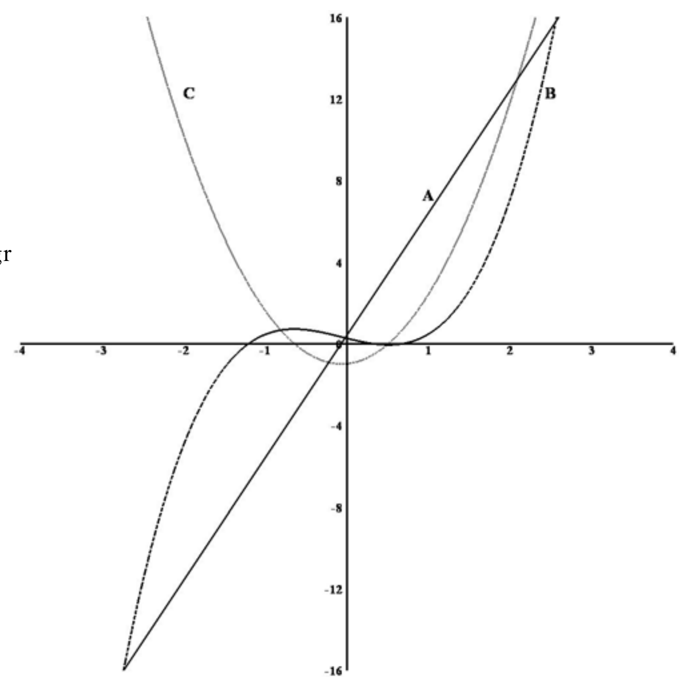
$\frac{dh}{dt}(t) = h'(t) = \mathbf{6.4t^{2.2} + 19.2t^{-4.2}}$ and $\left.\frac{dh(t)}{dt}\right|_{t=3} = h'(3) = \mathbf{71.94}$

$\frac{d^2h}{dt^2}(t) = h''(t) = \mathbf{14.08t^{1.2} - 80.64t^{-5.2}}$ and

$\left.\frac{d^2h(t)}{dt^2}\right|_{t=3} = h''(3) = \mathbf{52.35}$

12. The grid on the right contains graphs of three different functions, $f(x)$, $\frac{df}{dx}(x) = f'(x)$, $\frac{d^2f}{dx^2}(x) = f''(x)$. Indicate below which gr

(A solid black; B dashed black; C dotted grey) represents:
We know that wherever the slope of a tangent line to a graph of f is 0 (*i.e.*, where f has a horizontal tangent line), the derivative of f, f', will have a zero and its graph will therefore have an x-intercept. Similarly, wherever the slope of a tangent line to a graph of f' is 0, the derivative of f', f'', will have a zero and its graph will therefor have an x-intercept. However, wherever a graph of f changes concavity (called having a "point of inflection"), a graph of f'' will have an x-intercept. Looking at the given graphs we note that where graph B has horizontal tangents (around $x = -0.7$, 0.6), graph C has x-intercepts. We note also that where graph C has a horizontal tangent ($x = -0.1$), graph A has an x-intercept. Finally, where graph A has its x-intercept, graph B appears



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to be changing concavity. Thus, graph B is of *f*, graph C is of *f* ′, and graph A is of *f* ″.

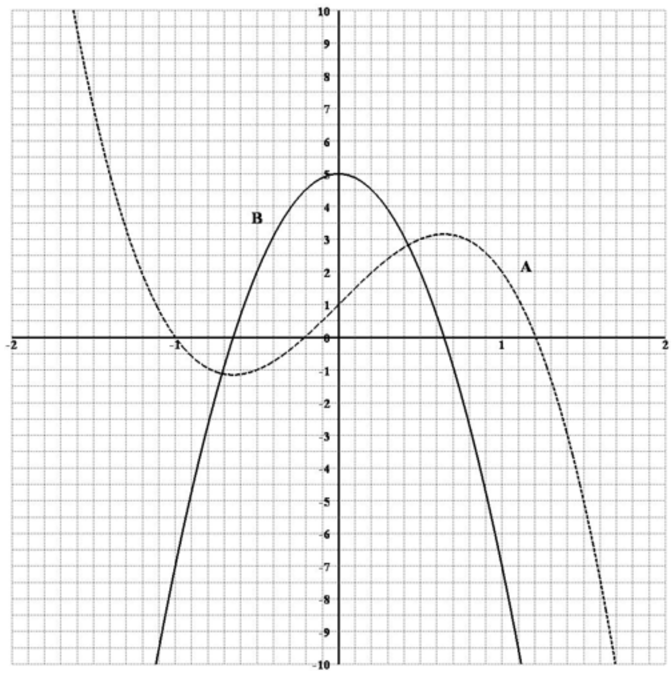
Graph B is the graph of the function $f(x)$

Graph C is the graph of the first derivative, function $\frac{df}{dx}(x) = f'(x)$

Graph A is the graph of the second derivative, function $\frac{d^2 f}{dx^2}(x) = f''(x)$

Note: We could also have derived these answers simply by noting that graph B appears to be of a cubic or degree three polynomial, graph C appears to be of a quadratic or degree two polynomial and graph A appears to be of a linear function or a degree one polynomial. Since the derivative of any polynomial is typically a polynomial of one degree less than the original polynomial, then graph B must be of *f*, graph C of *f* ′ and graph A of *f* ″.

13. The grid on the right shows the graph of $\frac{df}{dx}(x) = f'(x)$, the first derivative of a function $f(x)$ (A dashed black), and $\frac{d^2 f}{dx^2}(x) = f''(x)$, the second derivative of $f(x)$ (B solid black). Using interval notation answer each of the questions below:



a. The interval(s) on which $f(x)$ is increasing:

(-∞,-1)∪(-0.2,1.2)

The function *f* is increasing on any interval where the derivative *f* ′ is positive. Looking at the dashed black curve, A, we note that it is positive on **(-∞,-1)** and **(-0.2,1.2)**. Thus, these intervals are where *f* is increasing.

b. The interval(s) on which $f(x)$ is decreasing:

(-1,-0.2)∪(1.2,∞)

The function *f* is decreasing on any interval where the derivative *f* ′ is negative. Looking at the dashed black curve, A, we note that it is negative on **(-1,-0.2)** and **(1.2,∞)**. Thus, these intervals are where *f* is decreasing.

c. The interval(s) on which $f(x)$ is concave up: **(-0.65,0.65)**

The function *f* is concave up on any interval where the second derivative *f* ″ is positive. Looking at the solid black curve, B, we note that it is positive on **(-0.65,0.65)**. Thus, this interval is where *f* is concave up.

d. The interval(s) on which $f(x)$ is concave down: **(-∞,-0.65)∪(0.65,∞)**

The function *f* is concave down on any interval where the second derivative *f* ″ is negative. Looking at the solid black curve, B, we note that it is negative on **(-∞,-0.65)** and **(0.65,∞)**. Thus, these intervals are where *f* is concave down.

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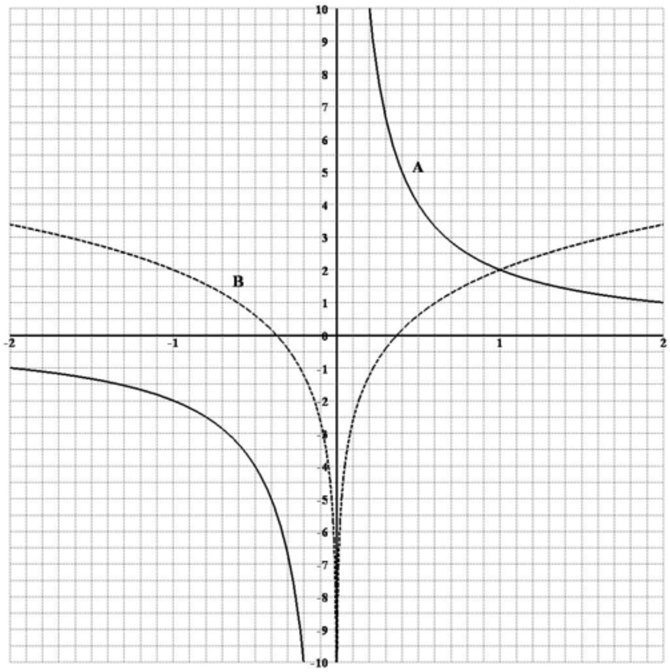
14. The grid on the right shows the graph of $\frac{df}{dx}(x)= f'(x)$, the first derivative of a function $f(x)$ (B dashed black), and $\frac{d^2 f}{dx^2}(x)= f''(x)$,the second derivative of $f(x)$ (A solid black). Using interval notation answer each of the questions below:



a. The interval(s) on which $f(x)$ is increasing:
(-∞,-0.36)∪(0.36,∞)

The function *f* is increasing on any interval where the derivative f' is positive. Looking at the dashed black curve, B, we note that it is positive on **(-∞,-0.36)** and **(0.36,∞)**. Thus, these intervals are where *f* is increasing.

b. The interval(s) on which $f(x)$ is decreasing:
(-0.36,0)∪(0,0.36)

The function *f* is decreasing on any interval where the derivative f' is negative. Looking at the dashed black curve, B, we note that it is negative on **(-0.36,0)** and **(0,0.36)**. Thus, these intervals are where *f* is decreasing.

c. The interval(s) on which $f(x)$ is concave up: **(0,∞)**

The function *f* is concave up on any interval where the second derivative f'' is positive. Looking at the solid black curve, A, we note that it is positive on **(0,∞)**. Thus, this interval is where *f* is concave up.

d. The interval(s) on which $f(x)$ is concave down: **(-∞,0)**

The function *f* is concave down on any interval where the second derivative f'' is negative. Looking at the solid black curve, A, we note that it is negative on **(-∞,0)**. Thus, this interval is where *f* is concave down.

15. AAA Widget Manufacturing has determined that cost of widget production, *C*, as a function of the number of widgets produced, *n*, (measured in thousands of widgets), is given by the formula $C(n)=\frac{0.3n^2+30}{n}$. Compute the following:

marginal cost at $n = 5$: **-0.9**
marginal cost at $n = 10$: **0**
marginal cost at $n = 20$: **0.225**
general formula for marginal cost, $C'(n)$: $\mathbf{0.3 - 30n^{-2}}$

$$C'(n)=\frac{dC(n)}{dn}=\frac{d\left(\frac{0.3n^2+30}{n}\right)}{dn}=\frac{d\left(0.3n+30n^{-1}\right)}{dn} \quad \textbf{(Rewriting to make Calculus easier)}$$

$$=\frac{d\left(0.3n\right)}{dn}+\frac{d\left(30n^{-1}\right)}{dn} \quad \textbf{(Sum Rule)}$$

$$=0.3\frac{d\left(n\right)}{dn}+30\frac{d\left(n^{-1}\right)}{dn} \quad \textbf{(Constant Multiple Rule)}$$

$$=0.3(1)+30\left(-1n^{-1-1}\right) \quad \textbf{(Identity \& Power Rules)}$$

$$=0.3-30n^{-2} \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$C'(5)=0.3-30(5)^{-2}=0.3-\frac{30}{25}=0.3-1.2=-0.9$$

$$C'(10)=0.3-30(10)^{-2}=0.3-\frac{30}{100}=0.3-0.3=0$$

$$C'(20)=0.3-30(20)^{-2}=0.3-\frac{30}{400}=0.3-0.075=0.225$$

Appendix E - Common Assignment Problems

Paper Assignment #7 with Solutions

1. Let $f(x) = 7e^x - 6x^5 + 21$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(7e^x - 6x^5 + 21\right)}{dx} = \frac{d\left(7e^x\right)}{dx} - \frac{d\left(6x^5\right)}{dx} + \frac{d\left(21\right)}{dx} \quad \textbf{(Sum \& Difference Rules)}$$

$$= 7\frac{d\left(e^x\right)}{dx} - 6\frac{d\left(x^5\right)}{dx} + 0 \quad \textbf{(Constant Multiple \& Constant Rules)}$$

$$= 7e^x - 6\left(5x^{5-1}\right) \quad \textbf{(e}^{(\ )}\textbf{ \& Power Rules)}$$

$$= 7e^x - 30x^4 \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

The first derivative of $f(x)$: $\frac{df(x)}{dx} = f'(x) = \mathbf{7e^x - 30x^4}$

2. Let $f(x) = 5x^3 - 5e^x$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(5x^3 - 5e^x\right)}{dx} = \frac{d\left(5x^3\right)}{dx} - \frac{d\left(5e^x\right)}{dx} \quad \textbf{(Difference Rule)}$$

$$= 5\frac{d\left(x^3\right)}{dx} - 5\frac{d\left(e^x\right)}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$= 5\left(3x^{3-1}\right) - 5e^x \quad \textbf{(Power \& e}^{(\ )}\textbf{ Rules)}$$

$$= 15x^2 - 5e^x \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'(3) = \left.\frac{df(x)}{dx}\right|_{x=3} = 15(3)^2 - 5e^3 = 135 - 5e^3$$

The first derivative of $f(x)$: $\frac{df(x)}{dx} = f'(x) = \mathbf{15x^2 - 5e^x}$ and $\left.\frac{df(x)}{dx}\right|_{x=3} = f'(3) = \mathbf{135 - 5e^3}$

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d\left(\frac{df(x)}{dx}\right)}{dx} = \frac{d\left(15x^2 - 5e^x\right)}{dx} = \frac{d\left(15x^2\right)}{dx} - \frac{d\left(5e^x\right)}{dx} \quad \textbf{(Difference Rule)}$$

$$= 15\frac{d\left(x^2\right)}{dx} - 5\frac{d\left(e^x\right)}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$= 15\left(2x^{2-1}\right) - 5e^x \quad \textbf{(Power \& e}^{(\ )}\textbf{ Rules)}$$

$$= 30x - 5e^x \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f''(3) = \left.\frac{d^2 f(x)}{dx^2}\right|_{x=3} = 30(3) - 5e^3 = 90 - 5e^3$$

The second derivative of $f(x)$: $\frac{d^2 f(x)}{dx^2} = f''(x) = \mathbf{30x - 5e^x}$ and $\left.\frac{d^2 f(x)}{dx^2}\right|_{x=3} = f''(3) = \mathbf{90 - 5e^3}$

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3. Let $f(x) = x^4 + 7e^x$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(x^4 + 7e^x\right)}{dx} = \frac{d\left(x^4\right)}{dx} + \frac{d\left(7e^x\right)}{dx} \quad \textbf{(Sum Rule)}$$

$$= \left(4x^{4-1}\right) + 7\frac{d\left(e^x\right)}{dx} \quad \textbf{(Power \& Constant Multiple Rules)}$$

$$= 4x^3 + 7e^x \quad \textbf{(e}^{(\;)}\textbf{ Rule)}$$

$$f'(1) = \left.\frac{df(x)}{dx}\right|_{x=1} = 4(1)^3 + 7e^1 = 4 + 7e$$

$\frac{df(x)}{dx} = f'(x) = \mathbf{4x^3 + 7e^x}$ and $\left.\frac{df(x)}{dx}\right|_{x=1} = f'(1) = \mathbf{4 + 7e}$

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d\left(\frac{df(x)}{dx}\right)}{dx} = \frac{d\left(4x^3 + 7e^x\right)}{dx} = \frac{d\left(4x^3\right)}{dx} + \frac{d\left(7e^x\right)}{dx} \quad \textbf{(Sum Rule)}$$

$$= 4\frac{d\left(x^3\right)}{dx} + 7\frac{d\left(e^x\right)}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$= 4\left(3x^{3-1}\right) + 7e^x \quad \textbf{(Power \& e}^{(\;)}\textbf{ Rules)}$$

$$= 12x^2 + 7e^x \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f''(1) = \left.\frac{d^2 f(x)}{dx^2}\right|_{x=1} = 12(1)^2 + 7e^1 = 12 + 7e$$

$\frac{d^2 f(x)}{dx^2} = f''(x) = \mathbf{12x^2 + 7e^x}$ and $\left.\frac{d^2 f(x)}{dx^2}\right|_{x=1} = f''(1) = \mathbf{12 + 7e}$

4. Let $f(x) = 6e^x$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(6e^x\right)}{dx} = 6\frac{d\left(e^x\right)}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$= 6e^x \quad \textbf{(e}^{(\;)}\textbf{ Rule)}$$

$$f'(3) = \left.\frac{df(x)}{dx}\right|_{x=3} = 6e^3$$

$\frac{df(x)}{dx} = f'(x) = \mathbf{6e^x}$ and $\left.\frac{df(x)}{dx}\right|_{x=3} = f'(3) = \mathbf{6e^3}$

We already know the slope of the requested tangent line, $m = f'(3) = 6e^3$. To determine an equation of the tangent line we also need a point on the tangent line. Since the point of tangency lies on the curve f, and all points on that curve have the form $(x, f(x))$, and we know that $x = 3$, the point of tangency is $(3, f(3)) = (3, 6e^3)$. Then, using the point-slope form of the equation of a line we have:

$$\frac{y - y_0}{x - x_0} = m \Rightarrow \frac{y - 6e^3}{x - 3} = 6e^3 \Leftrightarrow y - 6e^3 = 6e^3(x - 3) \Leftrightarrow y = 6e^3 x - 12e^3$$

Use the above to determine the equation of the tangent line to a graph of f at the point $x = 3$. The equation of this tangent line can be written in the form $y = mx + b$, where m is: $\mathbf{6e^3}$ and b is: $\mathbf{-12e^3}$

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5. Let $f(x) = (x^3 + 2x + 3)^4$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(\left(x^3+2x+3\right)^4\right)}{dx} = \frac{d\left(\left(x^3+2x+3\right)^4\right)}{d\left(x^3+2x+3\right)} \times \frac{d\left(x^3+2x+3\right)}{dx} \quad \text{(Chain Rule)}$$

$$= \left(4\left(x^3+2x+3\right)^{4-1}\right) \times \left[\frac{d\left(x^3\right)}{dx} + \frac{d\left(2x\right)}{dx} + \frac{d\left(3\right)}{dx}\right] \quad \text{(Power \& Sum Rules)}$$

$$= \left(4\left(x^3+2x+3\right)^3\right) \times \left[\left(3x^{3-1}\right) + 2\frac{d\left(x\right)}{dx} + 0\right] \quad \text{(Power, Constant Multiple \& Constant Rules)}$$

$$= \left(4\left(x^3+2x+3\right)^3\right) \times \left[3x^2 + 2(1)\right] \quad \text{(Identity Rule)}$$

$$= 4\left[3x^2+2\right]\left(x^3+2x+3\right)^3 \quad \text{(Identity Rule)}$$

$$f'(2) = \left.\frac{df(x)}{dx}\right|_{x=2} = 4\left[3(5)^2+2\right]\left((5)^3+2(5)+3\right)^3 = 4[77](138)^3 = 809{,}446{,}176$$

$\frac{df(x)}{dx} = f'(x) =$ **$4[3x^2 + 2](x^3 + 2x + 3)^3$** and $\left.\frac{df(x)}{dx}\right|_{x=5} = f'(5) =$ **809,446,176**

6. Let $f(x) = (5x + 4)^{-3}$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(\left(5x+4\right)^{-3}\right)}{dx} = \frac{d\left(\left(5x+4\right)^{-3}\right)}{d\left(5x+4\right)} \times \frac{d\left(5x+4\right)}{dx} \quad \text{(Chain Rule)}$$

$$= \left(-3\left(5x+4\right)^{-3-1}\right) \times \left[\frac{d\left(5x\right)}{dx} + \frac{d\left(4\right)}{dx}\right] \quad \text{(Power \& Sum Rules)}$$

$$= \left(-3\left(5x+4\right)^{-4}\right) \times \left[5\frac{d\left(x\right)}{dx} + 0\right] \quad \text{(Power, Constant Multiple \& Constant Rules)}$$

$$= \left(-3\left(5x+4\right)^{-4}\right) \times \left[5(1)\right] \quad \text{(Identity Rule)}$$

$$= -15\left(5x+4\right)^{-4} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'(2) = \left.\frac{df(x)}{dx}\right|_{x=2} = -15\left(5(2)+4\right)^{-4} = \frac{-15}{14^4} = \frac{-15}{38416} \doteq -0.0003904623074$$

$\frac{df(x)}{dx} = f'(x) =$ **$-15(5x + 4)^{-4}$** and $\left.\frac{df(x)}{dx}\right|_{x=2} = f'(2) =$ **-0.0003904623074**

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7. Let $f(x)=\sqrt{5x+4}$. Use the **rules of differentiation** to compute:

$$f'(x)=\frac{df(x)}{dx}=\frac{d\left(\sqrt{5x+4}\right)}{dx}=\frac{d\left((5x+4)^{1/2}\right)}{dx}=\frac{d\left((5x+4)^{1/2}\right)}{d\left(5x+4\right)}\times\frac{d\left(5x+4\right)}{dx}\quad \textbf{(Chain Rule)}$$

$$=\left(\frac{1}{2}(5x+4)^{1/2-1}\right)\times\left[\frac{d\left(5x\right)}{dx}+\frac{d\left(4\right)}{dx}\right]\quad \textbf{(Power \& Sum Rules)}$$

$$=\left(\frac{1}{2}(5x+4)^{-1/2}\right)\times\left[5\frac{d\left(x\right)}{dx}+0\right]\quad \textbf{(Power, Constant Multiple \& Constant Rules)}$$

$$=\left(\frac{1}{2}(5x+4)^{-1/2}\right)\times\left[5(1)\right]\quad \textbf{(Identity Rule)}$$

$$=\frac{5}{2}(5x+4)^{-1/2}\quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'(2)=\left.\frac{df(x)}{dx}\right|_{x=2}=\frac{5}{2}\left(5(2)+4\right)^{-1/2}=\frac{5}{2(14)^{1/2}}=\frac{5}{2\sqrt{14}}\doteq 0.6681531049$$

$\frac{df(x)}{dx}= f'(x) =$ **(5/2)(5x + 4)$^{-1/2}$** and $\left.\frac{df(x)}{dx}\right|_{x=4}= f'(4) =$ **0.6681531049**

8. Let $f(x)$ = -4ln(5x). Use the **rules of differentiation** to compute:

$$f'(x)=\frac{df(x)}{dx}=\frac{d\left(-4\ln(5x)\right)}{dx}=-4\frac{d\left(\ln(5x)\right)}{dx}\quad \textbf{(Constant Multiple Rule)}$$

$$=-4\frac{d\left(\ln(5x)\right)}{d(5x)}\times\frac{d\left(5x\right)}{dx}\quad \textbf{(Chain Rule)}$$

$$=-4\frac{1}{(5x)}\times 5\frac{d\left(x\right)}{dx}\quad \textbf{(ln( ) \& Constant Multiple Rules)}$$

$$=\frac{-4}{\cancel{5}x}\times\cancel{5}(1)\quad \textbf{(Identity Rule)}$$

$$=\frac{-4}{x}\quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'(4)=\left.\frac{df(x)}{dx}\right|_{x=4}=\frac{-4}{4}=-1$$

$\frac{df(x)}{dx}= f'(x) =$ **(-4/x)** and $\left.\frac{df(x)}{dx}\right|_{x=4}= f'(4) =$ **-1**

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9. Let $f(x) = [\ln(x)]^2$. Use the **rules of differentiation** to compute:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left([\ln(x)]^2\right)}{dx} = \frac{d\left([\ln(x)]^2\right)}{d[\ln(x)]} \times \frac{d(\ln(x))}{dx} \quad \text{(Chain Rule)}$$

$$= 2[\ln(x)]^{2-1} \times \frac{1}{x} \quad \text{(Power \& ln( ) Rules)}$$

$$= \frac{2\ln(x)}{x} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'(e^3) = \left.\frac{df(x)}{dx}\right|_{x=e^3} = \frac{2\ln(e^3)}{e^3} = \frac{2\times 3}{e^3} = \frac{6}{e^3} \doteq 0.2987224103$$

$\frac{df(x)}{dx} = f'(x) =$ **(2ln(x)/x)** and $\left.\frac{df(x)}{dx}\right|_{x=e^3} = f'(e^3) =$ **0.2987224103**

10. Given the table below

$(\ )$	-2	-1	0	1	2
$f(\ )$	2	1	-1	0	1
$\frac{df(\ )}{d(\ )}$	-1	-2	1	2	0
$g(\ )$	0	2	1	-2	-1
$\frac{dg(\ )}{d(\ )}$	2	1	-1	0	-2

compute:

$\left.\frac{df(g(\ ))}{d(\ )}\right|_{(\ )=1} =$ **0** $\qquad \left.\frac{df(g(\ ))}{d(\ )}\right|_{(\ )=-1} =$ **0**

$\left.\frac{dg(f(\ ))}{d(\ )}\right|_{(\ )=2} =$ **0** $\qquad \left.\frac{df(f(\ ))}{d(\ )}\right|_{(\ )=0} =$ **-2**

Solution:

$$\frac{df(g(\ ))}{d(\ )} = \frac{df(g(\ ))}{dg(\ )} \times \frac{dg(\ )}{d(\ )} \quad \text{(Chain Rule)}$$

$$\left.\frac{df(g(\ ))}{d(\ )}\right|_{(\ )=1} = \left.\frac{df(g(\ ))}{dg(\ )}\right|_{\substack{(\ )=1\\ g(\ )=g(1)}} \times \left.\frac{dg(\ )}{d(\ )}\right|_{(\ )=1}$$

$$= f'(g(1)) \times g'(1) = f'(-2) \times 0 = -1 \times 0 = 0$$

$$\frac{dg(f(\ ))}{d(\ )} = \frac{dg(f(\ ))}{df(\ )} \times \frac{df(\ )}{d(\ )} \quad \text{(Chain Rule)}$$

$$\left.\frac{dg(f(\ ))}{d(\ )}\right|_{(\ )=2} = \left.\frac{dg(f(\ ))}{df(\ )}\right|_{\substack{(\ )=2\\ f(\ )=f(2)}} \times \left.\frac{df(\ )}{d(\ )}\right|_{(\ )=2}$$

$$= g'(f(2)) \times f'(2) = g'(1) \times 0 = 0 \times 0 = 0$$

$$\frac{df(g(\ ))}{d(\ )} = \frac{df(g(\ ))}{dg(\ )} \times \frac{dg(\ )}{d(\ )} \quad \text{(Chain Rule)}$$

$$\left.\frac{df(g(\ ))}{d(\ )}\right|_{(\ )=-1} = \left.\frac{df(g(\ ))}{dg(\ )}\right|_{\substack{(\ )=-1\\ g(\ )=g(-1)}} \times \left.\frac{dg(\ )}{d(\ )}\right|_{(\ )=-1}$$

$$= f'(g(-1)) \times g'(-1) = f'(2) \times 1 = 0 \times 1 = 0$$

$$\frac{df(f(\ ))}{d(\ )} = \frac{df(f(\ ))}{df(\ )} \times \frac{df(\ )}{d(\ )} \quad \text{(Chain Rule)}$$

$$\left.\frac{df(f(\ ))}{d(\ )}\right|_{(\ )=0} = \left.\frac{df(f(\ ))}{df(\ )}\right|_{\substack{(\ )=0\\ f(\ )=f(0)}} \times \left.\frac{df(\ )}{d(\ )}\right|_{(\ )=0}$$

$$= f'(f(0)) \times f'(0) = f'(-1) \times 1 = -2 \times 1 = -2$$

Appendix E - Common Assignment Problems

11. The AAA Widget Manufacturing Corporation has modelled the demand curve for their Widgets, with demand D in thousands of widgets as a function of price p, in \$ per widget, based on their last five years of experience: $D(p)=\frac{8}{p}+1$. AAAWMC recently moved to a "Just In Time" widget manufacturing process, allowing them to produce exactly the number of widgets ordered, as they are ordered, with the cost of production C a function of the number of widgets, n, (in thousands) as given by $C(n)=11-e^{-n/4}$. The CEO of AAAWMC has just come into your office and demanded that you determine the derivative of cost of production C, as a function of price, p, *i.e.*, $\frac{dC\left(D(p)\right)}{dp}$.

Solutions:
There are two methods of doing this problem.
Method 1: We note that $\frac{dC\left(D(p)\right)}{dp}=\frac{dC\left(D(p)\right)}{dD(p)}\times\frac{dD(p)}{dp}$ by the Chain Rule. We will compute each of the two derivatives separately.

$$\frac{dC(n)}{dn}=\frac{d\left(11-e^{(-n/4)}\right)}{dn}=\frac{d\,(11)}{dn}-\frac{d\left(e^{(-n/4)}\right)}{dn}\quad\textbf{(Difference Rule)}$$

$$=0-\frac{d\left(e^{(-n/4)}\right)}{d\left(-n/4\right)}\times\frac{d\left(-n/4\right)}{dn}\quad\textbf{(Constant \& Chain Rules)}$$

$$=-e^{(-n/4)}\times\left(\frac{-1}{4}\right)\frac{d\,(n)}{dn}\quad\textbf{(e}^{(\ )}\textbf{ \& Constant Multiple Rules)}$$

$$=-e^{(-n/4)}\times\left(\frac{-1}{4}\right)(1)\quad\textbf{(Identity Rule)}$$

$$=\frac{e^{(-n/4)}}{4}\quad\textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\frac{dC(\ )}{d(\ )}=\frac{e^{(-(\ )/4)}}{4}\ \text{ so }\ \frac{dC(D(p))}{d(D(p))}=\frac{e^{(-(D(p))/4)}}{4}$$

$$\frac{dD(p)}{dp}=\frac{d\left(\frac{8}{p}+1\right)}{dp}=\frac{d\left(8p^{-1}+1\right)}{dp}=\frac{d\left(8p^{-1}\right)}{dp}+\frac{d\,(1)}{dp}\quad\textbf{(Sum Rule)}$$

$$=8\frac{d\left(p^{-1}\right)}{dp}+0\quad\textbf{(Constant Multiple \& Constant Rules)}$$

$$=8\left(-1p^{-1-1}\right)\quad\textbf{(Power Rule)}$$

$$=-8p^{-2}\quad\textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

Combining these two results we obtain:

$$\frac{dC\left(D(p)\right)}{dp}=\frac{dC\left(D(p)\right)}{dD(p)}\times\frac{dD(p)}{dp}=\frac{e^{(-(D(p))/4)}}{\not{4}}\times\left(-\overset{2}{\not{8}}\,p^{-2}\right)=\frac{-2e^{\left(-\left(8p^{-1}+1\right)/4\right)}}{p^{2}}$$

Method 2: We begin by replacing n in $C(n)$ with $D(p)$ so as to obtain $C(D(p))$.

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$$C(n)=\left(11-e^{\left(-n/4\right)}\right) \text{ and } D(p)=8p^{-1}+1 \text{ so } C\left(D(p)\right)=\left(11-e^{\left(-D(p)/4\right)}\right)=\left(11-e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)$$

Now we differentiate:

$$\frac{dC(p)}{dp}=\frac{d\left(11-e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)}{dp}=\frac{d(11)}{dp}-\frac{d\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)}{dp} \quad \text{(Difference Rule)}$$

$$=0-\frac{d\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)}{d\left(-\left(8p^{-1}+1\right)/4\right)}\times\frac{d\left(-\left(8p^{-1}+1\right)/4\right)}{dp} \quad \text{(Constant \& Chain Rules)}$$

$$=-\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)\times\left(\frac{-1}{4}\right)\frac{d\left(8p^{-1}+1\right)}{dp} \quad \text{(e}^{(\ )}\text{ \& Constant Multiple Rules)}$$

$$=\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)\times\left(\frac{1}{4}\right)\left[\frac{d\left(8p^{-1}\right)}{dp}+\frac{d(1)}{dp}\right] \quad \text{(Sum Rule)}$$

$$=\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)\times\left(\frac{1}{4}\right)\left[8\frac{d\left(p^{-1}\right)}{dp}+0\right] \quad \text{(Constant Multiple \& Constant Rules)}$$

$$=\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)\times\left(\frac{1}{4}\right)\left[8\left(-1p^{-1-1}\right)\right] \quad \text{(Power Rule)}$$

$$=\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right)\times\left(\frac{1}{\cancel{4}}\right)\left[\overset{2}{\cancel{8}}\left(-1p^{-2}\right)\right]=\frac{-2}{p^{2}}\left(e^{\left(-\left(8p^{-1}+1\right)/4\right)}\right) \quad \text{(Arith./Alg./Fns. Cleanup)}$$

We note that the answer is the same, no matter which method is used.

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12. The rate of photosynthesis in a leaf as a function of time, t, in days since the budding of the leaf is given by: $P(t) = 250\left(e^{-0.03t} - e^{-0.2t}\right)$.

$$P'(t) = \frac{dP(t)}{dt} = \frac{d\left(250\left(e^{-0.03t} - e^{-0.2t}\right)\right)}{dt} = 250\frac{d\left(e^{-0.03t} - e^{-0.2t}\right)}{dt} \quad \textbf{(Constant Multiple Rule)}$$

$$= 250\left[\frac{d\left(e^{-0.03t}\right)}{dt} - \frac{d\left(e^{-0.2t}\right)}{dt}\right] \quad \textbf{(Difference Rule)}$$

$$= 250\left[\frac{d\left(e^{(-0.03t)}\right)}{d(-0.03t)}\times\frac{d(-0.03t)}{dt} - \frac{d\left(e^{(-0.2t)}\right)}{d(-0.2t)}\times\frac{d(-0.2t)}{dt}\right] \quad \textbf{(Chain Rule)}$$

$$= 250\left[e^{(-0.03t)}\times(-0.03)\frac{d(t)}{dt} - e^{(-0.2t)}\times(-0.2)\frac{d(t)}{dt}\right] \quad \textbf{(e}^{(\ )}\textbf{ \& Constant Multiple Rules)}$$

$$= 250\left[e^{(-0.03t)}\times(-0.03)(1) - e^{(-0.2t)}\times(-0.2)(1)\right] \quad \textbf{(Identity Rule)}$$

$$= 250\left[-0.03e^{(-0.03t)} + 0.2e^{(-0.2t)}\right] \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

Determine the derivative function: $P'(t) = \frac{dP(t)}{dt} = 250\left[-0.03e^{(-0.03t)} + 0.2e^{(-0.2t)}\right]$

$$P'(t) = \frac{dP(t)}{dt} = 250\left[-0.03e^{(-0.03t)} + 0.2e^{(-0.2t)}\right] = 0$$

$$\Leftrightarrow \left[-0.03e^{(-0.03t)} + 0.2e^{(-0.2t)}\right] = 0 \Leftrightarrow 0.2e^{(-0.2t)} = 0.03e^{(-0.03t)} \Leftrightarrow \frac{e^{(-0.2t)}}{e^{(-0.03t)}} = \frac{0.03}{0.2}$$

$$\Leftrightarrow e^{-0.2t-(-0.03t)} = 0.15 \Leftrightarrow e^{-0.17t} = 0.15 \Leftrightarrow \ln\left(e^{-0.17t}\right) = \ln(0.15) \Leftrightarrow -0.17t = \ln(0.15)$$

$$\Leftrightarrow t = \frac{\ln(0.15)}{-0.17} \doteq 11.1595$$

Solve the equation $P'(t) = 0$ for the value of t at which the photosynthesis is at its maximum in the leaf:
$t \doteq$ **11.1595**

13. A couple of software engineers have an idea for a website unlike anything that has been seen before. Like the developers of YouTube, they are hoping that their concept will take off and after a year or so Google or Microsoft will buy them out for $1.65 billion and they can retire and enjoy life. Their business plan, such as it is (what can you expect from a pair of engineers who never took a management course in their life), is to double the number of "hits" to their site every 30 days. In the first 30 day period ($t = 0$) they have 1000 hits, and in the second 30 day period ($t = 1$) they have 2343 hits. Assume that growth continues exponentially based on this data. They plan to put their site up for sale when the **rate of growth of hits to their site in a 30 day period** reaches 1 billion.
    After how many months (30 day periods) will this happen? **18**
    Do they get to retire after a year? (Yes or No) **No**

Solution:
The sentence "Assume that growth continues exponentially based on this data." tells us that we can use an exponential model for the number of hits, H, as a function of time, t: $H(t) = Ca^t$.
The phrase "In the first 30 day period ($t = 0$) they have 1000 hits" tells us that $1000 = H(0) = Ca^0 = C$, so that our model is now $H(t) = 1000a^t$.
The phrase "in the second 30 day period ($t = 1$) they have 2343 hits" tells us that $2343 = H(1) = 1000a^1$, so $a = 2.343$, and our model is now $H(t) = 1000(2.343^t)$
The phrase "**rate of growth of hits**" indicates that we must compute $H'(t)$ so:

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$$H'(t) = \frac{dH(t)}{dt} = \frac{d\left(1000\left(2.343^t\right)\right)}{dt} = 1000\frac{d\left(2.343^t\right)}{dt} \quad \text{(Constant Multiple Rule)}$$

$$= 1000\left(2.343^t \ln(2.343)\right) \quad \text{(a}^{(\ )}\text{ Rule)}$$

The phrase "when the **rate of growth of hits to their site in a 30 day period** reaches 1 billion" tells us that we are interested in the value of t when $1000(\ln(2.343))2.343^t = 1{,}000{,}000{,}000$.
Solving this equation we obtain:

$$\cancel{1000}\ln(2.343)\left(2.343^t\right) = 1{,}000{,}000{,}\cancel{000} \Leftrightarrow \ln(2.343)\left(2.343^t\right) = 1{,}000{,}000$$

$$\Leftrightarrow \left(2.343^t\right) = \frac{1{,}000{,}000}{\ln(2.343)} \Leftrightarrow \ln\left(2.343^t\right) = \ln\left(\frac{1{,}000{,}000}{\ln(2.343)}\right) \Leftrightarrow t\ln\left(2.343\right) = \ln\left(\frac{1{,}000{,}000}{\ln(2.343)}\right)$$

$$\Leftrightarrow t = \frac{\ln\left(\frac{1{,}000{,}000}{\ln(2.343)}\right)}{\ln\left(2.343\right)} \doteq 16.4$$

We note that $t = 0$ represented "after 1 month", and $t = 1$ represented "after 2 months", so $t \doteq 16.4$ means their condition for sale and retirement will only be met after 18 months (or 17.4 to be more precise). Thus, they do not get to retire after one year.

14. Jeremy, just 25, is a recent Bachelor of Commerce graduate. His wealthy father gives him $1,000,000 as capital to start whatever business he would like. Jeremy is very cautious, realizing that this is gift is also a test by his father. He decides that he wants to invest the money in something that has slow long term growth, so that over a long period of years it will safely increase in value. After sifting through alternatives he decides that long term investment in real estate is his best choice. After poring through textbooks he discovers a theory that the value of a real estate investment, $V(t)$, is a logarithmic function of time t (measured in periods of 10 years): $V(t) = P\log_{10}(t + 10)$, where P is the initial investment made. What marginal value can Jeremy project for this investment after 30 years ($t = 3$)?

Solution:
Marginal value just means the derivative of value so our first step in solving this problem involves computing the derivative of V.

$$V'(t) = \frac{dV(t)}{dt} = \frac{d\left(1{,}000{,}000\log_{10}(t+10)\right)}{dt} = 1{,}000{,}000\frac{d\left(\log_{10}(t+10)\right)}{dt} \quad \text{(Constant Multiple Rule)}$$

$$= 1{,}000{,}000\frac{d\left(\log_{10}(t+10)\right)}{d(t+10)} \times \frac{d(t+10)}{dt} \quad \text{(Chain Rule)}$$

$$= 1{,}000{,}000\frac{1}{(t+10)\ln(10)} \times \left[\frac{d(t)}{dt} + \frac{d(10)}{dt}\right] \quad \text{(}\log_{10}(\ )\text{ \& Sum Rules)}$$

$$= 1{,}000{,}000\frac{1}{(t+10)\ln(10)} \times [1+0] \quad \text{(Identity \& Constant Rules)}$$

$$= \frac{1{,}000{,}000}{(t+10)\ln(10)} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$V'(3) = \left.\frac{dV(t)}{dt}\right|_{t=3} = \frac{1{,}000{,}000}{((3)+10)\ln(10)} = \frac{1{,}000{,}000}{13\ln(10)} \doteq 33407.27$$

Marginal value of investment after 30 years ($t = 3$): **33407.27**

Appendix E - Common Assignment Problems

15. The term T, in years, of a \$120,000 home mortgage, where the interest is 10%, can be approximated by $T(x)=\dfrac{5.315}{-6.8+\ln(x)}$,

where x is the monthly payment in dollars. Use the model to approximate the term of a home mortgage for which the monthly payment is \$1,190: $T(1190)$ = **18.87**

$$T(x)=\frac{5.315}{-6.8+\ln(x)} \Rightarrow T(1190)=\frac{5.315}{-6.8+\ln(1190)} \doteq 18.87$$

What is the total amount paid? **\$269,463.60**

The total amount paid would be \$1,190×18.87×12 ≐ \$269,463.60 (on a loan of \$120,000).

Determine the instantaneous rate of change (derivative) of T with respect to x when x = 1190:

$$T'(x)=\frac{dT(x)}{dx}=\frac{d\left(\dfrac{5.315}{-6.8+\ln(x)}\right)}{dx}=\frac{d\left(5.315\left(-6.8+\ln(x)\right)^{-1}\right)}{dx} \quad \textbf{(Algebra rewrite)}$$

$$=5.315\frac{d\left(-6.8+\ln(x)\right)^{-1}}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$=5.315\frac{d\left(-6.8+\ln(x)\right)^{-1}}{d\left(-6.8+\ln(x)\right)}\times\frac{d\left(-6.8+\ln(x)\right)}{dx} \quad \textbf{(Chain Rule)}$$

$$=5.315(-1)\left(-6.8+\ln(x)\right)^{-1-1}\times\left[\frac{d(-6.8)}{dx}+\frac{d\ln(x)}{dx}\right] \quad \textbf{(Power \& Sum Rule)}$$

$$=-5.315\left(-6.8+\ln(x)\right)^{-2}\times\left[0+\frac{1}{x}\right] \quad \textbf{(Constant \& ln( ) Rule)}$$

$$=\frac{-5.315\left(-6.8+\ln(x)\right)^{-2}}{x} \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$T'(1190)=\left.\frac{dT(x)}{dx}\right|_{x=1190}=\frac{-5.315\left(-6.8+\ln(1190)\right)^{-2}}{1190} \doteq -0.05628$$

Is this rate of change positive or negative? **negative**

Would higher monthly payments be beneficial? (Yes/No)

If the rate of change of T is negative (which it is), this means that as the payment increases, the term decreases. One does not need Calculus to predict this result. Of course, "beneficial" would mean that one ends up paying less in total. A quick calculation with x = 1191 or 1200 shows that increasing the monthly payment leads to a decrease in total payment. If we want to use Calculus to verify this we should compute the derivative of total payment, not term, and if that derivative is negative then an increased monthly payment would lead to a lower total payment.

Let $C(x) = T(x)\times x\times 12$, or total payments. To compute this derivative would require use of either the Quotient or Product Rule of differentiation so it is a good example for the next assignment.

Appendix E - Common Assignment Problems

Paper Assignment #8 with Solutions

1. If $g(x) = (2x^2 - 5x + 9)e^x$, then determine $g'(x) = \frac{dg(x)}{dx}$ = $\mathbf{(2x^2 - x + 4)\, e^x}$

Solution:

$$g'(x) = \frac{dg(x)}{dx} = \frac{d\left((2x^2-5x+9)e^x\right)}{dx}$$
$$= \left(\frac{d\left((2x^2-5x+9)\right)}{dx}\right)e^x + (2x^2-5x+9)\left(\frac{d\left(e^x\right)}{dx}\right) \quad \text{(Product Rule)}$$
$$= \left(\frac{d\left(2x^2\right)}{dx} - \frac{d\left(5x\right)}{dx} + \frac{d(9)}{dx}\right)e^x + (2x^2-5x+9)\left(e^x\right) \quad \text{(Sum, Difference \& e}^{(\ )}\text{ Rules)}$$
$$= \left(2\frac{d\left(x^2\right)}{dx} - 5\frac{d\left(x\right)}{dx} + 0\right)e^x + (2x^2-5x+9)e^x \quad \text{(Constant Multiple \& Constant Rules)}$$
$$= \left(2(2x) - 5(1)\right)e^x + (2x^2-5x+9)e^x \quad \text{(Power \& Identity Rules)}$$
$$= \left(4x-5\right)e^x + (2x^2-5x+9)e^x \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$= \left((4x-5) + (2x^2-5x+9)\right)e^x \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$= \left(2x^2 - x + 4\right)e^x \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

2. If $f(x) = 3x^4 \ln(x)$, then determine $f'(x) = \frac{df(x)}{dx}$ = $\mathbf{3x^3[1 + 4\ln(x)]}$ and $f'\left(e^2\right) = \left.\frac{df(x)}{dx}\right|_{x=e^2}$ = $\mathbf{27e^6}$

Solution:

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(3x^4\ln(x)\right)}{dx} = 3\frac{d\left(x^4\ln(x)\right)}{dx} \quad \text{(Constant Multiple Rule)}$$
$$= 3\left[\left(\frac{d\left(x^4\right)}{dx}\right)\ln(x) + x^4\left(\frac{d\left(\ln(x)\right)}{dx}\right)\right] \quad \text{(Product Rule)}$$
$$= 3\left[\left(4x^{4-1}\right)\ln(x) + x^{\not{4}\,3}\left(\frac{1}{\not{x}}\right)\right] \quad \text{(Power \& ln(\ \ ) Rules)}$$
$$= 3\left[4x^3\ln(x) + x^3\right] \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$= 3x^3\left[1 + 4\ln(x)\right] \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$f'\left(e^2\right) = \left.\frac{df(x)}{dx}\right|_{x=e^2} = \left.\left(3x^3\left[1+4\ln(x)\right]\right)\right|_{x=e^2} = 3\left(e^2\right)^3\left[1 + 4\not{\ln}\left(\not{e}^2\right)\right]$$
$$= 3e^6\left[1+4(2)\right] = 3e^6\left[9\right] = 27e^6$$

Appendix E - Common Assignment Problems

3. If $f(x)=\dfrac{2x+5}{3x+7}$ then determine $f'(x)=\dfrac{df(x)}{dx}$ = **-(3x + 7)$^{-2}$** and $f'(-1)=\left.\dfrac{df(x)}{dx}\right|_{x=-1}$ = **-0.0625**

Solutions:

$$f'(x)=\frac{df(x)}{dx}=\frac{d\left(\frac{2x+5}{3x+7}\right)}{dx}=\frac{\left(\frac{d(2x+5)}{dx}\right)(3x+7)-(2x+5)\left(\frac{d(3x+7)}{dx}\right)}{(3x+7)^2}\quad\textbf{(Quotient Rule)}$$

$$=\frac{\left(\frac{d(2x)}{dx}+\frac{d(5)}{dx}\right)(3x+7)-(2x+5)\left(\frac{d(3x)}{dx}+\frac{d(7)}{dx}\right)}{(3x+7)^2}\quad\textbf{(Sum Rule)}$$

$$=\frac{\left(2\frac{d(x)}{dx}+0\right)(3x+7)-(2x+5)\left(3\frac{d(x)}{dx}+0\right)}{(3x+7)^2}\quad\textbf{(Constant Multiple \& Constant Rules)}$$

$$=\frac{(2(1))(3x+7)-(2x+5)(3(1))}{(3x+7)^2}\quad\textbf{(Identity Rule)}$$

$$=\frac{2(3x+7)-3(2x+5)}{(3x+7)^2}=\frac{6x+14-6x-15}{(3x+7)^2}\quad\textbf{(Arith./Alg./Functions Cleanup)}$$

$$=\frac{-1}{(3x+7)^2}=-(3x+7)^{-2}\quad\textbf{(Arith./Alg./Functions Cleanup)}$$

$$f'(-1)=\left.\frac{df(x)}{dx}\right|_{x=-1}=-(3(-1)+7)^{-2}=-(4)^{-2}=\frac{-1}{16}=-0.0625$$

4. Given $g(y)=\dfrac{e^y}{2-5y}$, determine $g'(y)=\dfrac{dg(y)}{dy}=\dfrac{[7-5y]\,e^y}{(2-5y)^2}$

Solution:

$$g'(y)=\frac{dg(y)}{dy}=\frac{d\left(\frac{e^y}{2-5y}\right)}{dy}=\frac{\left(\frac{de^y}{dy}\right)(2-5y)-e^y\left(\frac{d(2-5y)}{dy}\right)}{(2-5y)^2}\quad\textbf{(Quotient Rule)}$$

$$=\frac{e^y(2-5y)-e^y\left(\frac{d(2)}{dy}-\frac{d(5y)}{dy}\right)}{(2-5y)^2}\quad\textbf{(e}^{(\ )}\textbf{ \& Difference Rules)}$$

$$=\frac{e^y(2-5y)-e^y\left(0-5\frac{d(y)}{dy}\right)}{(2-5y)^2}\quad\textbf{(Constant \& Constant Multiple Rules)}$$

$$=\frac{e^y(2-5y)+e^y(5(1))}{(2-5y)^2}\quad\textbf{(Identity Rule)}$$

$$=\frac{e^y\left[(2-5y)+(5)\right]}{(2-5y)^2}=\frac{[7-5y]\,e^y}{(2-5y)^2}\quad\textbf{(Arithmetic/Algebra/Functions Rules)}$$

Appendix E - Common Assignment Problems

5. Given $h(z)=\dfrac{2z}{3+4z^2}$, determine $h'(z)=\dfrac{dh(z)}{dz}=2\left[\dfrac{3-4z^2}{\left(3+4z^2\right)^2}\right]$ and $h'(2)=\left.\dfrac{dh(z)}{dz}\right|_{z=2}=\dfrac{-26}{361}$

Use the latter result to determine an equation for the line tangent to the curve h at the point (2, h(2)). An equation of this tangent line can be written in the form $y = mz + b$ where $m \doteq$ **-0.07202** and $b \doteq$ **0.3546**

Solutions:

$$h'(z)=\frac{dh(z)}{dz}=\frac{d\left(\frac{2z}{3+4z^2}\right)}{dz}=2\frac{d\left(\frac{z}{3+4z^2}\right)}{dz}\quad \textbf{(Constant Multiple Rule)}$$

$$=2\left[\frac{\left(\frac{dz}{dz}\right)\left(3+4z^2\right)-z\left(\frac{d\left(3+4z^2\right)}{dz}\right)}{\left(3+4z^2\right)^2}\right]\quad \textbf{(Quotient Rule)}$$

$$=2\left[\frac{(1)\left(3+4z^2\right)-z\left(\frac{d(3)}{dz}+\frac{d\left(4z^2\right)}{dz}\right)}{\left(3+4z^2\right)^2}\right]\quad \textbf{(Identity \& Sum Rules)}$$

$$=2\left[\frac{\left(3+4z^2\right)-z\left(0+4\frac{d\left(z^2\right)}{dz}\right)}{\left(3+4z^2\right)^2}\right]\quad \textbf{(Constant \& Constant Multiple Rules)}$$

$$=2\left[\frac{\left(3+4z^2\right)-z\left(4(2z)\right)}{\left(3+4z^2\right)^2}\right]\quad \textbf{(Power Rule)}$$

$$=2\left[\frac{3+4z^2-8z^2}{\left(3+4z^2\right)^2}\right]=2\left[\frac{3-4z^2}{\left(3+4z^2\right)^2}\right]\quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$h'(2)=\left.\frac{dh(z)}{dz}\right|_{z=2}=2\left[\frac{3-4(2)^2}{\left(3+4(2)^2\right)^2}\right]=2\left[\frac{3-16}{(3+16)^2}\right]=2\left[\frac{-13}{(19)^2}\right]=\frac{-26}{361}$$

$$h(z)=\frac{2z}{3+4z^2}\Rightarrow h(2)=\frac{2(2)}{3+4(2)^2}=\frac{4}{3+16}=\frac{4}{19}$$

$$\frac{y-\frac{4}{19}}{x-2}=\frac{-26}{361}\Leftrightarrow y-\frac{4}{19}=\frac{-26}{361}(x-2)\Leftrightarrow y=\frac{-26}{361}x+\frac{128}{361}\Leftrightarrow y\doteq -0.07202x+0.3546$$

Appendix E - Common Assignment Problems

6. If $f(x)=\dfrac{2\sin(x)}{3+4\cos(x)}$ then determine $f'(x)=\dfrac{df(x)}{dx}=2\left[\dfrac{3\cos(x)+4}{(3+4\cos(x))^2}\right]$ and $f'\left(\dfrac{\pi}{4}\right)=\left.\dfrac{df(x)}{dx}\right|_{x=\frac{\pi}{4}}$ = **0.3604**

Solutions:

$$f'(x)=\frac{df(x)}{dx}=\frac{d\left(\dfrac{2\sin(x)}{3+4\cos(x)}\right)}{dx}=2\frac{d\left(\dfrac{\sin(x)}{3+4\cos(x)}\right)}{dx}\quad \text{(Constant Multiple Rule)}$$

$$=2\left[\frac{\left(\dfrac{d\sin(x)}{dx}\right)(3+4\cos(x))-\sin(x)\left(\dfrac{d(3+4\cos(x))}{dx}\right)}{(3+4\cos(x))^2}\right]\quad \text{(Quotient Rule)}$$

$$=2\left[\frac{\cos(x)(3+4\cos(x))-\sin(x)\left(\dfrac{d(3)}{dx}+\dfrac{d(4\cos(x))}{dx}\right)}{(3+4\cos(x))^2}\right]\quad \text{(Sine \& Sum Rules)}$$

$$=2\left[\frac{3\cos(x)+4\cos^2(x)-\sin(x)\left(0+4\dfrac{d\cos(x)}{dx}\right)}{(3+4\cos(x))^2}\right]\quad \text{(Constant \& Constant Multiple Rules)}$$

$$=2\left[\frac{3\cos(x)+4\cos^2(x)-\sin(x)(4(-\sin(x))}{(3+4\cos(x))^2}\right]\quad \text{(Cosine Rule)}$$

$$=2\left[\frac{3\cos(x)+4\cos^2(x)+4\sin^2(x)}{(3+4\cos(x))^2}\right]\quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$=2\left[\frac{3\cos(x)+4\left(\cos^2(x)+\sin^2(x)\right)}{(3+4\cos(x))^2}\right]\quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$=2\left[\frac{3\cos(x)+4}{(3+4\cos(x))^2}\right]\quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f'\left(\frac{\pi}{4}\right)=\left.\frac{df(x)}{dx}\right|_{x=\frac{\pi}{4}}=2\left[\frac{3\cos\left(\frac{\pi}{4}\right)+4}{\left(3+4\cos\left(\frac{\pi}{4}\right)\right)^2}\right]=2\left[\frac{\left(\frac{3}{\sqrt{2}}\right)+4}{\left(3+\left(\frac{4}{\sqrt{2}}\right)\right)^2}\right]\doteq 0.3604$$

Appendix E - Common Assignment Problems

7. If $f(x)=\sin\left(x^9\right)$ then determine $f'(x)=\frac{df(x)}{dx}=9x^8\cos\left(x^9\right)$ and $f'(2)=\left.\frac{df(x)}{dx}\right|_{x=2}\doteq -2296.7041$

Solutions:

$$f'(x)=\frac{df(x)}{dx}=\frac{d\sin\left(x^9\right)}{dx}=\frac{d\sin\left(x^9\right)}{d\left(x^9\right)}\times\frac{d\left(x^9\right)}{dx}\quad\text{(Chain Rule)}$$
$$=\cos\left(x^9\right)\times 9\left(x^{9-1}\right)\quad\text{(Sine \& Power Rules)}$$
$$=\cos\left(x^9\right)\times 9\left(x^8\right)=9x^8\cos\left(x^9\right)\quad\text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$f'(2)=\left.\frac{df(x)}{dx}\right|_{x=2}=9(2)^8\cos\left((2)^9\right)=9(256)\cos(512)\doteq -2296.7041$$

8. If $f(x)=6\left(e^{x\sin(x)}\right)$ then determine $f'(x)=\frac{df(x)}{dx}=6\left[\sin(x)+x\cos(x)\right]\left(e^{(x\sin(x))}\right)$

Solution:

$$f'(x)=\frac{df(x)}{dx}=\frac{d6\left(e^{x\sin(x)}\right)}{dx}=6\frac{d\left(e^{x\sin(x)}\right)}{dx}\quad\text{(Constant Multiple Rule)}$$
$$=6\frac{d\left(e^{(x\sin(x))}\right)}{d\left(x\sin(x)\right)}\times\frac{d\left(x\sin(x)\right)}{dx}\quad\text{(Chain Rule)}$$
$$=6\left(e^{(x\sin(x))}\right)\left[\frac{d\left(x\right)}{dx}\sin(x)+x\frac{d\left(\sin(x)\right)}{dx}\right]\quad\text{(Product Rule)}$$
$$=6\left(e^{(x\sin(x))}\right)\left[(1)\sin(x)+x\cos(x)\right]\quad\text{(Identity \& Cosine Rules)}$$
$$=6\left[\sin(x)+x\cos(x)\right]\left(e^{(x\sin(x))}\right)\quad\text{(Arithmetic/Algebra/Functions Cleanup)}$$

9. Determine an equation of the line tangent to the curve $f(x) = 3\tan(x)$ at the point $(-\pi/3, f(-\pi/4))$. An equation for this line can be written in the form $y = mx + b$ where m = ____ and b = ____.

Solutions:

There was an error in the question concerning the point at which the tangent line should be computed. The *x* and *y* coordinates do not match. Either it should be $(-\pi/3, f(-\pi/3))$ or $(-\pi/4, f(-\pi/4))$, but not $(-\pi/3, f(-\pi/4))$ as given. The solutions provided below use both values.

$$f'(x)=\frac{df(x)}{dx}=\frac{d3\tan(x)}{dx}=3\frac{d\tan(x)}{dx}\quad\text{(Constant Multiple Rule)}=3\sec^2(x)\quad\text{(Tangent Rule)}$$
$$f\left(-\pi/4\right)=3\tan\left(-\pi/4\right)=3(-1)=-3$$
$$f'\left(-\pi/4\right)=\left.\frac{df(x)}{dx}\right|_{x=-\pi/4}=3\sec^2\left(-\pi/4\right)=3\left(-\sqrt{2}\right)^2=3\times 2=6$$
$$\frac{y-(-3)}{x-\left(-\pi/4\right)}=6\Leftrightarrow y+3=6\left(x+\pi/4\right)\Leftrightarrow y=6x+\left(3\pi/2-3\right)$$
$$f'\left(-\pi/3\right)=\left.\frac{df(x)}{dx}\right|_{x=-\pi/3}=3\sec^2\left(-\pi/3\right)=3\left(2\right)^2=3\times 4=12$$
$$\frac{y-\left(-\sqrt{3}\right)}{x-\left(-\pi/3\right)}=12\Leftrightarrow y+\sqrt{3}=12\left(x+\pi/3\right)\Leftrightarrow y=12x+\left(4\pi-\sqrt{3}\right)$$

Thus, at $(-\pi/3, f(-\pi/3))$ we have m = 12 and b = $\left(4\pi-\sqrt{3}\right)$

and at $(-\pi/4, f(-\pi/4))$ we have m = 6 and b = $\left(3\pi/2-3\right)$

Appendix E - Common Assignment Problems

10. If $g(y) = 7\ln(\sin(x))$ then determine $g'(y) = \frac{dg(y)}{dy}$ = **7cot(*y*)**

Solution:

Clearly there was an error here in terms of the independent variable, which should be *y* not *x*. The calculation below corrects that error.

$$g'(y) = \frac{dg(y)}{dy} = \frac{d7\ln(\sin(y))}{dy} = 7\frac{d\ln(\sin(y))}{dy} \quad \text{(Constant Multiple Rule)}$$

$$= 7\frac{d\ln(\sin(y))}{d\sin(y)} \times \frac{d\sin(y)}{dy} \quad \text{(Chain Rule)}$$

$$= 7\frac{1}{\sin(y)} \times \cos(y) \quad \text{(ln( ) and Sine Rules)}$$

$$= 7\frac{\cos(y)}{\sin(y)} = 7\cot(y) \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

11. Given $f(x) = \cos(x)$, determine the 73rd derivative of $f(x)$:

Solution:

If we have to do this one step for each derivative this will take 73 steps so there must be a pattern that we can use to simplify the process. Otherwise, this is an unreasonable problem. Thus we begin the process but watch for a pattern.

$f'(x) = (\cos(x))' = -\sin(x)$; $f''(x) = (-\sin(x))' = -(\sin(x))' = -\cos(x)$; $f'''(x) = (-\cos(x))' = -(\cos(x))' = -(-\sin(x)) = \sin(x)$;
$f^{(4)}(x) = (\sin(x))' = \cos(x)$.

Hopefully it is clear that since $f^{(4)}(x) = f(x)$, then $f^{(5)}(x) = f'(x)$, $f^{(6)}(x) = f''(x)$, $f^{(7)}(x) = f'''(x)$, and $f^{(8)}(x) = f^{(4)}(x)$. More generally, we can see that the process of differentiating the trigonometric function $\cos(x)$ is “cyclic”, with a “period of length 4”. That is, every four derivatives the cycle repeats. Thus, if we are asked for the 73rd derivative, we just have to figure out where we would be in that cycle when we reach the 73rd derivative. Now $73 = 4\times18+1$. Thus, the 73rd derivative, would be just the same as the first, *i.e.*, $f'(x) = -\sin(x)$.

Appendix E - Common Assignment Problems

12. Given $-5x^2 + 8xy + 6y^3 = -125$, determine $y' = \frac{dy}{dx} = \frac{5x-4y}{4x+9y^2}$ and the slope m of the tangent line to the given curve at the point (-5, 0): **1.25**

Solutions:

$$-5x^2 + 8xy + 6y^3 = -125$$

$$\frac{d\left(-5x^2 + 8xy + 6y^3\right)}{dx} = \frac{d(-125)}{dx} \quad \text{(Implicit Differentiation)}$$

$$\Leftrightarrow \frac{d\left(-5x^2\right)}{dx} + \frac{d(8xy)}{dx} + \frac{d\left(6y^3\right)}{dx} = 0 \quad \text{(Sum \& Constant Rules)}$$

$$\Leftrightarrow -5\frac{d\left(x^2\right)}{dx} + 8\frac{d(xy)}{dx} + 6\frac{d\left(y^3\right)}{dx} = 0 \quad \text{(Constant Multiple Rule)}$$

$$\Leftrightarrow -5\left(2x^{2-1}\right) + 8\left[\left(\frac{dx}{dx}\right)y + x\left(\frac{dy}{dx}\right)\right] + 6\left(\frac{dy^3}{dy}\right)\times\left(\frac{dy}{dx}\right) = 0 \quad \text{(Power, Product \& Chain Rules)}$$

$$\Leftrightarrow -10x + 8\left[(1)y + x\left(\frac{dy}{dx}\right)\right] + 6\left(3y^{3-1}\right)\times\left(\frac{dy}{dx}\right) = 0 \quad \text{(Identity \& Power Rules)}$$

$$\Leftrightarrow -10x + 8\left[y + x\left(\frac{dy}{dx}\right)\right] + 18y^2\left(\frac{dy}{dx}\right) = 0 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow -10x + 8y + 8x\left(\frac{dy}{dx}\right) + 18y^2\left(\frac{dy}{dx}\right) = 0 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow \left(8x + 18y^2\right)\left(\frac{dy}{dx}\right) = 10x - 8y \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow \left(\frac{dy}{dx}\right) = \frac{10x - 8y}{8x + 18y^2} = \frac{5x - 4y}{4x + 9y^2} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\left.\left(\frac{dy}{dx}\right)\right|_{(x,y)=(-5,0)} = \frac{5(-5) - 4(0)}{4(-5) + 9(0)^2} = \frac{5\cancel{(-5)}}{4\cancel{(-5)}} = \frac{5}{4} = 1.25$$

Appendix E - Common Assignment Problems

13. Given $xy^3 + xy = 18$, determine $y' = \frac{dy}{dx} = \frac{-y^3 - y}{3xy^2 + x}$ and the slope m of the tangent line to the given curve at the point (9, 1):

-0.05556

Solutions:

$$xy^3 + xy = 18$$

$$\frac{d\left(xy^3 + xy\right)}{dx} = \frac{d18}{dx} \quad \text{(Implicit Differentiation)}$$

$$\Leftrightarrow \frac{d\left(xy^3\right)}{dx} + \frac{d\left(xy\right)}{dx} = 0 \quad \text{(Sum \& Constant Rules)}$$

$$\Leftrightarrow \left[\left(\frac{dx}{dx}\right)y^3 + x\left(\frac{dy^3}{dx}\right)\right] + \left[\left(\frac{dx}{dx}\right)y + x\left(\frac{dy}{dx}\right)\right] = 0 \quad \text{(Product Rule)}$$

$$\Leftrightarrow \left[(1)y^3 + x\left(\frac{dy^3}{dy}\right)\times\left(\frac{dy}{dx}\right)\right] + \left[(1)y + x\left(\frac{dy}{dx}\right)\right] = 0 \quad \text{(Identity \& Chain Rules)}$$

$$\Leftrightarrow \left[y^3 + x\left(3y^{3-1}\right)\times\left(\frac{dy}{dx}\right)\right] + \left[y + x\left(\frac{dy}{dx}\right)\right] = 0 \quad \text{(Power Rule)}$$

$$\Leftrightarrow y^3 + 3xy^2\left(\frac{dy}{dx}\right) + y + x\left(\frac{dy}{dx}\right) = 0 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow \left(3xy^2 + x\right)\left(\frac{dy}{dx}\right) = -y^3 - y \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\Leftrightarrow \left(\frac{dy}{dx}\right) = \frac{-y^3 - y}{3xy^2 + x} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$\left(\frac{dy}{dx}\right)\Bigg|_{(x,y)=(9,1)} = \frac{-(1)^3 - (1)}{3(9)(1)^2 + (9)} = \frac{-2}{36} = \frac{-1}{18} \doteq -0.05556$$

Appendix E - Common Assignment Problems

14. Given $f(x)=x^{3\sin(2x)}$ then determine $f'(x)=\dfrac{df(x)}{dx}=3\left[2\cos(2x)\ln(x)+\dfrac{\sin(2x)}{x}\right]x^{3\sin(2x)}$

Solutions:
We note that the function *f* has the form of (variable base)$^{\text{(variable exponent)}}$. There is no rule for such a function. Instead we use the procedure called "logarithmic differentiation". The technique involves three major steps. First, we apply the ln( ) function to both sides of the function definition. The intention here is always the same, to use the property of logarithms, $\ln(a^b) = b\ln(a)$, to convert the form from its current state into a product, for which we have a differentiation rule. Second, now having an implicitly defined function we use implicit differentiation. When all derivative rules that are appropriate have been applied, and the only derivative left is $\dfrac{df(x)}{dx}$, we solve for that derivative, just as in other implicit differentiation examples. However, the third and final step, any $f(x)$ that appears on the right hand side must be replaced by the original value of $f(x)$ given at the beginning of the problem.
It is important to note when using this technique that many steps are identical across all problems, and only a few change, those that deal with the particular details of the "variable base" and the "variable exponent". Noting this helps us to follow the procedure more rapidly.

$$f(x)=x^{3\sin(2x)} \Rightarrow \ln(f(x))=\ln\left(x^{3\sin(2x)}\right) \Leftrightarrow \ln(f(x))=(3\sin(2x))\ln(x) \quad \text{(Log Differentiation)}$$

$$\frac{d\ln(f(x))}{dx}=\frac{d\,(3\sin(2x))\ln(x)}{dx} \quad \text{(Implicit Differentiation)}$$

$$\Leftrightarrow\left(\frac{d\ln(f(x))}{df(x)}\right)\times\left(\frac{df(x)}{dx}\right)=3\frac{d\left(\sin(2x)\ln(x)\right)}{dx} \quad \text{(Chain \& Constant Multiple Rules)}$$

$$\Leftrightarrow\frac{1}{f(x)}\times\left(\frac{df(x)}{dx}\right)=3\left[\left(\frac{d\left(\sin(2x)\right)}{dx}\right)\ln(x)+\sin(2x)\left(\frac{d\ln(x)}{dx}\right)\right] \quad \text{(ln( ) \& Product Rules)}$$

$$\Leftrightarrow\frac{1}{f(x)}\times\left(\frac{df(x)}{dx}\right)=3\left[\left(\frac{d\left(\sin(2x)\right)}{d2x}\times\frac{d2x}{dx}\right)\ln(x)+\sin(2x)\left(\frac{1}{x}\right)\right] \quad \text{(Chain \& ln( ) Rules)}$$

$$\Leftrightarrow\frac{1}{f(x)}\times\left(\frac{df(x)}{dx}\right)=3\left[\left(\cos(2x)\times2\left(\frac{dx}{dx}\right)\right)\ln(x)+\frac{\sin(2x)}{x}\right] \quad \text{(Sine \& Constant Multiple Rules)}$$

$$\Leftrightarrow\frac{1}{f(x)}\times\left(\frac{df(x)}{dx}\right)=3\left[\left(\cos(2x)\times2(1)\right)\ln(x)+\frac{\sin(2x)}{x}\right] \quad \text{(Identity Rule)}$$

$$\Leftrightarrow\left(\frac{df(x)}{dx}\right)=3\left[2\cos(2x)\ln(x)+\frac{\sin(2x)}{x}\right]f(x) \quad \text{(Identity Rule)}$$

$$\Leftrightarrow\left(\frac{df(x)}{dx}\right)=3\left[2\cos(2x)\ln(x)+\frac{\sin(2x)}{x}\right]x^{3\sin(2x)} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

Appendix E - Common Assignment Problems

15. A drug company has developed a new antibiotic drug that requires only a single time-release dose to eliminate Clostridium difficile (C. difficile) infection from a person's body. From experimental data they have a model for concentration, C, of the drug in a patient's blood as a function of time, t (in days): $C(t) = t\,(1+t)^{-0.8\,t}$. This function tracks the rise in drug concentration in the blood from an initial value of 0 (at time $t = 0$) to a maximum concentration, and then gradual elimination of the drug from the blood. Determine $C'(t) = \left(\frac{dC(t)}{dt}\right) = \left(\frac{1}{t} - 0.8\left[\ln\left((1+t)\right) + \frac{t}{(1+t)}\right]\right)t(1+t)^{-0.8t}$

Solution:
Once again we notice that we have a function that in part has the form (variable base)$^{\text{(variable exponent)}}$, and so again we must use the technique referred to as logarithmic differentiation.

$$C(t) = t(1+t)^{-0.8t} \Leftrightarrow \ln\left(C(t)\right) = \ln\left(t(1+t)^{-0.8t}\right) \Leftrightarrow \ln\left(C(t)\right) = \ln\left(t\right) - 0.8t\ln\left((1+t)\right) \quad \textbf{(Log. Diff.)}$$

$$\frac{d\ln\left(C(t)\right)}{dt} = \frac{d\ln\left(t\right)}{dt} - \frac{d\left(0.8t\ln\left((1+t)\right)\right)}{dt} \quad \textbf{(Implicit Differentiation)}$$

$$\Leftrightarrow \left(\frac{d\ln\left(C(t)\right)}{dC(t)}\right) \times \left(\frac{dC(t)}{dt}\right) = \frac{1}{t} - 0.8\frac{d\left(t\ln\left((1+t)\right)\right)}{dt} \quad \textbf{(Chain, ln( ) \& Constant Multiple Rules)}$$

$$\Leftrightarrow \left(\frac{1}{C(t)}\right)\left(\frac{dC(t)}{dt}\right) = \frac{1}{t} - 0.8\left[\left(\frac{dt}{dt}\right)\ln\left((1+t)\right) + t\left(\frac{d\ln(1+t)}{dt}\right)\right] \quad \textbf{(ln( ) \& Product Rules)}$$

$$\Leftrightarrow \left(\frac{1}{C(t)}\right)\left(\frac{dC(t)}{dt}\right) = \frac{1}{t} - 0.8\left[(1)\ln\left((1+t)\right) + t\left(\frac{d\ln(1+t)}{d(1+t)} \times \frac{d(1+t)}{dt}\right)\right] \quad \textbf{(Identity \& Chain Rules)}$$

$$\Leftrightarrow \left(\frac{1}{C(t)}\right)\left(\frac{dC(t)}{dt}\right) = \frac{1}{t} - 0.8\left[\ln\left((1+t)\right) + t\left(\frac{1}{(1+t)} \times \left[\frac{d1}{dt} + \frac{dt}{dt}\right]\right)\right] \quad \textbf{(ln( ) \& Sum Rules)}$$

$$\Leftrightarrow \left(\frac{1}{C(t)}\right)\left(\frac{dC(t)}{dt}\right) = \frac{1}{t} - 0.8\left[\ln\left((1+t)\right) + \frac{t}{(1+t)}[0+1]\right] \quad \textbf{(Constant \& Identity Rules)}$$

$$\Leftrightarrow \left(\frac{dC(t)}{dt}\right) = \left(\frac{1}{t} - 0.8\left[\ln\left((1+t)\right) + \frac{t}{(1+t)}\right]\right)C(t) \quad \textbf{(Arithmetic/Algebra/Function Cleanup)}$$

$$\Leftrightarrow \left(\frac{dC(t)}{dt}\right) = \left(\frac{1}{t} - 0.8\left[\ln\left((1+t)\right) + \frac{t}{(1+t)}\right]\right)t(1+t)^{-0.8t} \quad \textbf{(Arithmetic/Algebra/Function Cleanup)}$$

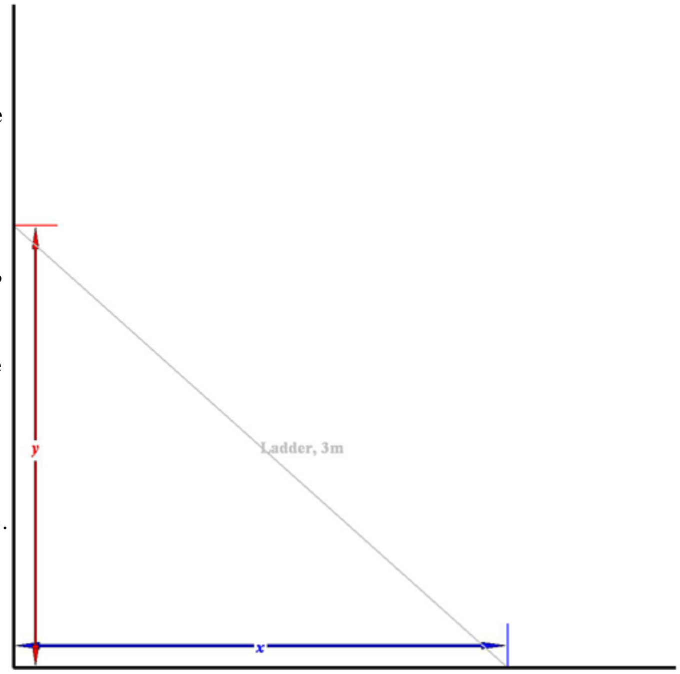
Appendix E - Common Assignment Problems

Paper Assignment #9 with Solutions

1. A ladder that is 3 metres long is leaning against a wall. The base of the ladder starts to slip away from the wall at a rate of 1.5 m/s. How fast is the top of the ladder sliding down the wall when the base is 1/3 metre from the wall?

Solution:

Step 0: Read the problem at least twice. Because we are given a rate of change ("The base of the ladder starts to slip away from the wall at a rate of 1.5 m/s."), and asked to determine a rate of change ("How fast is the top of the ladder sliding down the wall when the base is 1/3 metre from the wall?"), and the distance the base of the ladder is from the wall is "related" to the "height of the top of the ladder on the wall", this is a related rates problem.



Step 1: We draw a diagram to illustrate the problem. Those quantities that are changing in the problem (height of the top of the ladder on the wall, distance the base of the ladder is from the wall) **are labelled in the diagram with letters since they are "variable". Those quantities that cannot change in the context of the problem** (the length of the ladder) **can be labelled with the constant value. The purpose of the diagram is to extract one or more equations that "relate" the variable(s) whose rate was given to the variable(s) whose rate was requested. The diagram does not need to be artistic, just good enough to suggest the appropriate equation(s).**

Step 2: From the problem we extract the given rate(s) and the requested rate, and write them in derivative notation. Note that one must pay particular attention to the sign of any given rate(s). In words, the sign can be expressed in many ways. For example, in this problem the sentence ("The base of the ladder starts to slip away from the wall at a rate of 1.5 m/s.") tells us that the ladder slips **away** from the wall, so x must be increasing, so the derivative of x must be positive. Given $\frac{dx}{dt}=1.5$, determine $\left.\frac{dy}{dt}\right|_{x=1/3}$. Note that given the a fixed length ladder, if the bottom slips away from the wall, the top slides down the wall. Thus, we anticipate that the value of $\left.\frac{dy}{dt}\right|_{x=1/3}$ should turn out to be negative. If it doesn't, then we will have made an error somewhere.

Step 3: Extract one or more equations from the diagram, equations that connect the variable(s) whose rate(s) were given to the variable(s) whose rate(s) were requested. In this case we need to connect x to y.

Hopefully everyone noticed the right triangle formed by the ladder, wall and ground. Using Pythagoras' Theorem we obtain the equation: $x^2 + y^2 = 3^2$ (EQ #1). Note that this equation implicitly defines y as a function of x.

Step 4: If multiple equations were deduced, one may simplify first. With a single equation, one differentiates immediately, keeping in mind what derivative(s) one already knows and what derivative one is attempting to determine.

$$x^2+y^2=3^2 \Rightarrow \frac{d\left(x^2+y^2\right)}{dt}=\frac{d9}{dt} \Leftrightarrow \frac{dx^2}{dt}+\frac{dy^2}{dt}=0$$

$$\Leftrightarrow \frac{dx^2}{dx}\times\frac{dx}{dt}+\frac{dy^2}{dy}\times\frac{dy}{dt}=0 \Leftrightarrow 2x\times\frac{dx}{dt}+2y\times\frac{dy}{dt}=0 \quad \text{(EQ \#2)}$$

Step 5: After differentiating, we check to see what variable expressions we know values for and what we are attempting to determine. If there are variables or variable expressions that we do not know values for, other than the rate that we are attempting to determine, then we must think about how such variables or variable expressions can be determined. In this example, in EQ #2, we know $x = 1/3$, $\frac{dx}{dt}=1.5$, and $\left.\frac{dy}{dt}\right|_{x=1/3}$ is what we are trying to determine. The only difficulty is that we do not know the value of y. However, if we return to EQ #1 and substitute in $x = 1/3$, we can solve for y:

$$x^2+y^2=9 \Rightarrow \left(1/3\right)^2+y^2=9 \Leftrightarrow y^2=9-1/9=\frac{81-1}{9}=\frac{80}{9} \Leftrightarrow y=\pm\sqrt{\frac{80}{9}}=\pm\frac{4}{3}\sqrt{5}$$

Appendix E - Common Assignment Problems

We note that in the context of this problem, where *y* represents the height of the top of the ladder on the wall, *y* cannot be negative (top of ladder has dug its way underground?), so only the positive value for *y* makes sense in the context of the problem. Thus,

$$2x\times\frac{dx}{dt}+2y\times\frac{dy}{dt}=0\Rightarrow \not{2}\left(\tfrac{1}{\not{3}}\right)\times\frac{\not{3}}{\not{2}}+2\left(\frac{4}{3}\sqrt{5}\right)\times\frac{dy}{dt}=0\Leftrightarrow 1+2\left(\frac{4}{3}\sqrt{5}\right)\times\frac{dy}{dt}=0$$

$$\Leftrightarrow\left(\frac{8}{3}\sqrt{5}\right)\times\frac{dy}{dt}=-1\Leftrightarrow\frac{dy}{dt}=\frac{-1}{\left(\frac{8}{3}\sqrt{5}\right)}=\frac{-3}{\left(8\sqrt{5}\right)}\doteq -0.1677$$

Step 6: Always answer a problem posed in words with a sentence answer. Find the sentence in the question that asked the question, and turn into a statement using the numerical answer determined in calculation. Here the question was: "How fast is the top of the ladder sliding down the wall when the base is 1/3 metre from the wall?". We rephrase this as "When the base of the ladder is ⅓ metre from the wall the ladder is sliding down the wall at a rate of approximately 0.1677 m/s".
Note how our calculation gave us a negative answer for the derivative, just as we anticipated, so the value of *y* is becoming smaller. However, in our word answer, we indicate direction by saying that the ladder is sliding down the wall, hence we remove the minus sign on the number.

2. The population of New Hedonia is 250,000, and rising at the rate of 5,000 people per year. The total yearly personal income in New Hedonia is \$45,000,000, and rising at the rate of \$500,000 per year.
   What is the current per capita personal income? How fast is it rising or falling? Is it rising or falling?

Solution:
In this example, if we let *P* represent population of New Hedonia, *t* represent time (years), *T* represent total personal income, *I* represent per capita personal, then $I=\frac{T}{P}$ is the standard meaning of per capita personal income. The **current** per capita personal income would be:

$$I(0)=\frac{T(0)}{P(0)}=\frac{\$\,\overset{9}{\not{4}\not{5}},00\not{0},\not{0}\not{0}\not{0}}{\underset{5}{\not{2}\not{5}}\,\not{0},\not{0}\not{0}\not{0}\text{ people}}=\frac{900\$}{5\text{ person}}=180\frac{\$}{\text{person}}$$

Now we consider the remaining questions, which make this a related rate problem.
Step 0: Read the problem at least twice. Because we are given two rates of change ("The population of New Hedonia is 250,000, and rising at the rate of 5,000 people per year" and "The total yearly personal income in New Hedonia is \$45,000,000, and rising at the rate of \$500,000 per year"), and asked to determine a rate of change ("What is the current per capita personal income? How fast is it rising or falling? Is it rising or falling?"), and per capita personal income is "related" to both the population size and total yearly personal income, this is a related rates problem.
Step 1: We draw a diagram to illustrate the problem. The main purpose of the diagram is to help us extract an equation connecting the variable(s) whose rates were given and the variable(s) whose rates were requested. In this example we can generate that equation directly and no diagram is needed.
Step 2: From the problem we extract the given rate(s) and the requested rate, and write them in derivative notation. Note that one must pay particular attention to the sign of any given rate(s). In words, the sign can be expressed in many ways. In this example, again letting *P* represent population of New Hedonia, *t* represent time (years), *T* represent total personal income, *I* represent per capita personal income, we note that we have been given *P*(0) = 250,000, $\left.\frac{dP}{dt}\right|_{t=0}=5{,}000$, *T*(0) = 45,000,000 and $\left.\frac{dT}{dt}\right|_{t=0}=500{,}000$, and asked to determine $\left.\frac{dI}{dt}\right|_{t=0}=?$

Step 3: Extract one or more equations from the diagram, equations that connect the variable(s) whose rate(s) were given to the variable(s) whose rate(s) were requested. We don't have a diagram here, but have no need for one given that we know that $I=\frac{T}{P}$ is the standard meaning of per capita personal income.
Step 4: With a single equation, one differentiates immediately, keeping in mind what derivative(s) one already knows and what derivative one is attempting to determine.

Appendix E - Common Assignment Problems

$$I=\frac{T}{P}\Rightarrow\frac{dI}{dt}=\frac{d\left(\frac{T}{P}\right)}{dt}=\frac{\left(\frac{dT}{dt}\right)P-T\left(\frac{dP}{dt}\right)}{P^2}\quad\text{(Quotient Rule)}\quad \text{EQ \#1}$$

Step 5: After differentiating, we check to see what variable expressions we know values for and what we are attempting to determine. If there are variables or variable expressions that we do not know values for, other than the rate that we are attempting to determine, then we must think about how such variables or variable expressions can be determined. We note that in this example we know values for both rates and both variables on the right of equation #1, so when we substitute in that information we will have determined the rate on the left hand side of equation #1, which is the rate that we were requested to determine.

$$\frac{dI}{dt}=\frac{\left(\frac{dT}{dt}\right)P-T\left(\frac{dP}{dt}\right)}{P^2}\Rightarrow\left.\frac{dI}{dt}\right|_{t=0}=\frac{\left(\left.\frac{dT}{dt}\right|_{t=0}\right)P(0)-T(0)\left(\left.\frac{dP}{dt}\right|_{t=0}\right)}{P^2(0)}$$

$$\Leftrightarrow\left.\frac{dI}{dt}\right|_{t=0}=\frac{(500{,}000)(250{,}000)-(45{,}000{,}000)(5{,}000)}{(250{,}000)^2}$$

$$=\frac{(500{,}000)(250{,}000)-(180\times250{,}000)(5{,}000)}{(250{,}000)^2}$$

$$=\cancel{(250{,}000)}\left(\frac{(500{,}000)-180(5{,}000)}{(250{,}000)^{\cancel{2}}}\right)=\left(\frac{(500{,}000)-900{,}000}{(250{,}000)}\right)$$

$$=\left(\frac{-\overset{8}{\cancel{40}}\,\cancel{0},\cancel{0}\cancel{0}\cancel{0}}{(\underset{5}{\cancel{25}}\,\cancel{0},\cancel{0}\cancel{0}\cancel{0})}\right)=-\frac{8}{5}=-1.6$$

Step 6: Always answer a problem posed in words with a sentence answer. Find the sentence in the question that asked the question, and turn into a statement using the numerical answer determined in calculation. Here the questions that were asked were "How fast is it rising or falling?" and "Is it rising or falling?". Since the rate of change (derivative) that we computed is negative, then the per capita income must be falling (negative derivative ⇔ decreasing function). The per capita income is falling at a rate of 1.6 \$/person/year.

3. The business manager of an independent newspaper estimates that annual advertising revenue received by the newspaper will be $R(x) = 1.2x^2 + 6x + 137$ thousand dollars when its circulation is x thousand copies. The circulation of the newspaper is currently 100,000 and is increasing at a constant rate of 3.4 thousand copies per year. At what rate will the annual advertising revenue be increasing

   with respect to time 4 years from now? **947.376 thousands of dollars per year**

Solution:

Step 0: Read the problem at least twice. Because we are given a rate of change ("is increasing at a constant rate of 3.4 thousand copies per year"), and asked to determine a rate of change ("At what rate will the annual advertising revenue be increasing with respect to time 4 years from now?"), and annual advertising revenue is "related" to circulation, this is a related rates problem.

Step 1: We draw a diagram to illustrate the problem. The main purpose of the diagram is to help us extract an equation connecting the variable(s) whose rates were given and the variable(s) whose rates were requested. In this example we can generate that equation directly and no diagram is needed.

Step 2: From the problem we extract the given rate(s) and the requested rate, and write them in derivative notation. Note that one must pay particular attention to the sign of any given rate(s). In words, the sign can be expressed in many ways. In this example, we have been given $x(0) = 100$ thousand, $\frac{dx}{dt}=3.4\text{ thousand/year}$, and asked to determine $\left.\frac{dR}{dt}\right|_{t=4}=?$

Step 3: Extract one or more equations from the diagram, equations that connect the variable(s) whose rate(s) were given to the variable(s) whose rate(s) were requested. We don't have a diagram here, but have no need for one, given the equation connecting annual advertising revenue, R, to circulation, x, which is $R(x) = 1.2x^2 + 6x + 137$.

Step 4: With a single equation, one differentiates immediately, keeping in mind what derivative(s) one already knows and what derivative one is attempting to determine.

Appendix E - Common Assignment Problems

$$\frac{dR}{dt}=\frac{dR}{dx}\times\frac{dx}{dt}\text{ (Chain Rule) }=\frac{d\left(1.2x^2+6x+137\right)}{dx}\times\frac{dx}{dt}$$
$$=\left(\frac{d\left(1.2x^2\right)}{dx}+\frac{d(6x)}{dx}+\frac{d(137)}{dx}\right)\times\frac{dx}{dt}\text{ (Sum Rule)}$$
$$=\left(1.2\frac{d\left(x^2\right)}{dx}+6\frac{d(x)}{dx}+0\right)\times\frac{dx}{dt}\text{ (Constant Multiple \& Constant Rules)}$$
$$=\left(1.2(2x)+6(1)\right)\times\frac{dx}{dt}\text{ (Power \& Identity Rules)}$$
$$=\left(2.4x+6\right)\times\frac{dx}{dt}\text{ (Arithmetic/Algebra/Functions Cleanup)}$$

Step 5: After differentiating, we check to see what variable expressions we know values for and what we are attempting to determine. If there are variables or variable expressions that we do not know values for, other than the rate that we are attempting to determine, then we must think about how such variables or variable expressions can be determined. In this problem we have been given $\frac{dx}{dt}=3.4\text{ thousand/year}$, and $x(0)$ = 100 thousand. Since we were told that x is increasing at a **constant** rate, and only linear functions have a constant rate of change, this means that $x(t) = 3.4\,t + 100$. Thus, we can compute $x(4) = 3.4(4) + 100 = 113.6$. We note that we now have all unknowns from the equation above in Step 4, other than the derivative that we are attempting to determine. Substituting this information into the equation from Step 4, we solve:

$$\frac{dR}{dt}=\left(2.4x+6\right)\times\frac{dx}{dt}\Rightarrow\left.\frac{dR}{dt}\right|_{t=4}=(2.4\times113.6+6)\times3.4=947.376\text{ thousands/year}$$

Step 6: Always answer a problem posed in words with a sentence answer. Find the sentence in the question that asked the question, and turn into a statement using the numerical answer determined in calculation. In this problem the question sentence is “At what rate will the annual advertising revenue be increasing with respect to time 4 years from now?” We rewrite this as, in 4 years from now the annual advertising revenue will be increasing at a rate of 947.376 thousands of dollars per year.

4. The marketing department of a computer manufacturer estimates that the demand D (in thousands of units per year) for a laptop is related to price by $D(p) = 5100 - 0.7p$. Because of efficiency and technological advances, the prices are falling at a rate of 85 dollars per year. The current price of a

   laptop is $1,900. At what rate are the revenues falling: **$207,400/year**

Solution:

Step 0: Read the problem at least twice. Because we are given a rate of change (“the prices are falling at a rate of 85 dollars per year”), and asked to determine a rate of change (“At what rate are the revenues falling”), and revenue is “related” to price, this is a related rates problem.

Step 1: We draw a diagram to illustrate the problem. The main purpose of the diagram is to help us extract an equation connecting the variable(s) whose rates were given and the variable(s) whose rates were requested. In this example we can generate that equation directly and no diagram is needed.

Step 2: From the problem we extract the given rate(s) and the requested rate, and write them in derivative notation. Note that one must pay particular attention to the sign of any given rate(s). In words, the sign can be expressed in many ways. In this example, we have been given $p(0)$ = $1,900, $\frac{dp}{dt}=-85\text{ \$/year}$, and asked to determine $\left.\frac{dR}{dt}\right|_{t=0}=?$

Step 3: Extract one or more equations from the diagram, equations that connect the variable(s) whose rate(s) were given to the variable(s) whose rate(s) were requested. We don’t have a diagram here, but have no need for one, given the equation connecting revenue, R, to demand, D, which is $R = Dp = (5100 - 0.7p)p = 5100p - 0.7p^2$.

Step 4: With a single equation, one differentiates immediately, keeping in mind what derivative(s) one already knows and what derivative one is attempting to determine.

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$$\frac{dR}{dt}=\frac{dR}{dp}\times\frac{dp}{dt}\ \text{(Chain Rule)}=\frac{d\left(5100p-0.7p^2\right)}{dp}\times\frac{dp}{dt}$$

$$=\left(\frac{d\left(5100p\right)}{dp}-\frac{d\left(0.7p^2\right)}{dp}\right)\times\frac{dp}{dt}\ \text{(Difference Rule)}$$

$$=\left(5100\frac{d\left(p\right)}{dp}-0.7\frac{d\left(p^2\right)}{dp}\right)\times\frac{dp}{dt}\ \text{(Constant Multiple Rule)}$$

$$=\left(5100(1)-0.7(2p)\right)\times\frac{dp}{dt}\ \text{(Identity \& Power Rule)}$$

$$=\left(5100-1.4p\right)\times\frac{dp}{dt}\ \text{(Arithmetic/Algebra/Functions Cleanup)}$$

Step 5: After differentiating, we check to see what variable expressions we know values for and what we are attempting to determine. If there are variables or variable expressions that we do not know values for, other than the rate that we are attempting to determine, then we must think about how such variables or variable expressions can be determined. In this problem we have been given $\frac{dp}{dt}=-85\ \$/\text{year}$, and $p(0)$ = \$1,900. We note that we now have all unknowns from the equation above in Step 4, other than the derivative that we are attempting to determine. Substituting this information into the equation from Step 4, we solve:

$$\frac{dR}{dt}=\left(5{,}100-1.4p\right)\times\frac{dp}{dt}\Rightarrow\left.\frac{dR}{dt}\right|_{t=0}=\left(5{,}100-1.4p(0)\right)\times\left.\frac{dp}{dt}\right|_{t=0}$$

$$\Leftrightarrow\left.\frac{dR}{dt}\right|_{t=0}=\left(5{,}100-1.4\times1{,}900\right)\times(-85)=-207{,}400$$

Step 6: Always answer a problem posed in words with a sentence answer. Find the sentence in the question that asked the question, and turn into a statement using the numerical answer determined in calculation. In this problem the question sentence is “At what rate are the revenues falling” We rewrite this as, the revenues will be falling at a rate of 207,400 dollars per year. Note that the word falling takes the place of the “-” in the computed result.

5. The function $f(x) = (5x - 4)\,e^{3x}$ has one critical number, which is **7/15**.

Solution:

A critical number of a function f is a value of x at which $f\,'$ is either 0 or discontinuous. To determine critical numbers of a function f our first step is to compute $f\,'$.

$$f'(x)=\frac{df(x)}{dx}=\frac{d\left((5x-4)e^{(3x)}\right)}{dx}=\left(\frac{d(5x-4)}{dx}\right)\times e^{(3x)}+(5x-4)\times\left(\frac{de^{(3x)}}{dx}\right)\ \text{(Product Rule)}$$

$$=\left(\frac{d(5x)}{dx}-\frac{d(4)}{dx}\right)\times e^{(3x)}+(5x-4)\times\left(\frac{de^{(3x)}}{d(3x)}\times\frac{d(3x)}{dx}\right)\ \text{(Difference \& Chain Rule)}$$

$$=\left(5\frac{d(x)}{dx}-(0)\right)\times e^{(3x)}+(5x-4)\times\left(e^{(3x)}\times3\frac{d(x)}{dx}\right)\ \text{(Constant Multiple, Constant, \& e Rule)}$$

$$=\left(5(1)\right)\times e^{(3x)}+(5x-4)\times\left(e^{(3x)}\times3(1)\right)\ \text{(Identity Rule)}$$

$$=5e^{(3x)}+(5x-4)\left(3e^{(3x)}\right)=\left(5+3(5x-4)\right)e^{(3x)}=\left(5+15x-12\right)e^{(3x)}=\left(15x-7\right)e^{(3x)}$$

Note that in general we wish to write $f\,'$ as a product so that we can use the very basic concept that when $a \times b = 0$ then either $a = 0$ or $b = 0$. Thus, instead of looking at the whole product, $a \times b$, we can examine each factor, a and b, separately.

Thus, $f\,' = 0 \Leftrightarrow (15x - 7)e^{3x} = 0 \Leftrightarrow 15x - 7 = 0$ or $e^{3x} = 0 \Leftrightarrow 15x = 7$, but e^{3x} can never equal 0 (think of the graph of exponential growth functions) $\Leftrightarrow x = 7/15$.

Since both $(15x - 7)$ and e^{3x} are continuous functions, $f\,'$ is also continuous, so there are no critical numbers to be found by virtue of $f\,'$ having a discontinuity.

Thus, our only critical number is $x = 7/15$.

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6. The function $f(x) = -2x^3 + 33x^2 - 168x + 3$ has one local minimum and one local maximum.
   The local minimum occurs at x = **4** with y-value = **-269**.
   The local maximum occurs at x = **7** with y-value = **-242**.

Solution:
Step one is to determine f'.

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(-2x^3 + 33x^2 - 168x + 3\right)}{dx}$$

$$= \frac{d-2x^3}{dx} + \frac{d33x^2}{dx} - \frac{d168x}{dx} + \frac{d3}{dx} \quad \textbf{(Sum \& Difference Rules)}$$

$$= -2\frac{dx^3}{dx} + 33\frac{dx^2}{dx} - 168\frac{dx}{dx} + 0 \quad \textbf{(Constant Multiple \& Constant Rules)}$$

$$= -2\left(3x^2\right) + 33(2x) - 168(1) \quad \textbf{(Power \& Identity Rules)}$$

$$= -6x^2 + 66x - 168 \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$= -6\left(x^2 + 11x - 28\right) \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

$$= -6(x-4)(x-7) \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

Step two is to determine all critical numbers (values of x that make f' be 0 or discontinuous)
Since f' is a polynomial and continuous everywhere, there are no critical numbers due to f' being discontinuous.
$f' = 0 \Leftrightarrow -6(x - 4)(x - 7) = 0 \Leftrightarrow x = 4, 7$
Step three is to create a miniature table recording the critical numbers, the value of f at these numbers (called critical values), and the sign of f' on intervals around the critical numbers

		m		*M*	
x		4		7	
$f(x)$	↘ ↘ ↘	→ -269	↗ ↗ ↗	→ -242	↘ ↘ ↘
$f'(x)$	- - -	0	+ + +	0	- - -

$f(4) = -2(4)^3 + 33(4)^2 - 168(4) + 3 = -2(64) + 33(16) - 168(4) + 3 = -269$
$f(7) = -2(7)^3 + 33(7)^2 - 168(7) + 3 = -2(343) + 33(49) - 168(7) + 3 = -242$

To check the sign of f' we will use 0 to represent $(-\infty,4)$, 5 to represent $(4,7)$ and 8 to represent $(7,\infty)$.

$f'(0) = -\times-\times- = -$; $f'(5) = -\times+\times- = +$; $f'(8) = -\times+\times+ = -$

We then interpret the signs of f' as telling us the direction of f. That is, where f' is negative (-), f is decreasing (↘), and where f' is positive (+), f is increasing (↗).
From the table we can see that at $x = 4$, f has a local minimum, the value being -269, and at $x = 7$, f has a local maximum, the value being -242.

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7. The function $f(x) = 9x + 8x^{-1}$ has one local minimum and one local maximum.

The local minimum occurs at $x = \frac{2\sqrt{2}}{3}$ with y-value = $12\sqrt{2}$.

The local maximum occurs at $x = -\frac{2\sqrt{2}}{3}$ with y-value = $-12\sqrt{2}$.

Solution:
Step one is to determine f'.

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(9x+8x^{-1}\right)}{dx} = \frac{d9x}{dx} + \frac{d8x^{-1}}{dx} \quad \text{(Sum Rule)}$$

$$= 9\frac{dx}{dx} + 8\frac{dx^{-1}}{dx} \quad \text{(Constant Multiple Rule)}$$

$$= 9(1) + 8\left(-1 \times x^{-1-1}\right) \quad \text{(Identity \& Power Rules)}$$

$$= 9 - 8x^{-2} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$= 9 - \frac{8}{x^2} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$= \frac{9x^2}{x^2} - \frac{8}{x^2} = \frac{9x^2-8}{x^2} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

Step two is to determine all critical numbers (values of x that make f' be 0 or discontinuous)
Since f' is a rational function and continuous everywhere the denominator is not zero, so the only place where f' is discontinuous is at $x = 0$.

$$f'(x) = 0 \Leftrightarrow 9x^2 - 8 = 0 \Leftrightarrow 9x^2 = 8 \Leftrightarrow x^2 = \frac{8}{9} \Leftrightarrow x = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}$$

Step three is to create a miniature table recording the critical numbers, the value of f at these numbers (called critical values), and the sign of f' on intervals around the critical numbers

		M		VA		m	
x		$-\frac{2\sqrt{2}}{3}$		0		$\frac{2\sqrt{2}}{3}$	
$f(x)$	↗ ↗ ↗	→ $-12\sqrt{2}$	↘ ↘ ↘ ↓	U	↑ ↘ ↘ ↘	$12\sqrt{2}$ →	↗ ↗ ↗
$f'(x)$	+ + +	0	- - -	U	- - -	0	+ + +

$$f\left(-\frac{2\sqrt{2}}{3}\right) = \overset{3}{\cancel{9}}\left(-\frac{2\sqrt{2}}{\cancel{3}}\right) + 8\left(-\frac{2\sqrt{2}}{3}\right)^{-1} = -6\sqrt{2} - \overset{2\sqrt{2}}{\cancel{8}}\left(\frac{3}{\cancel{2\sqrt{2}}}\right) = -12\sqrt{2}$$

$$f(0) = 9(0) + 8(0)^{-1} = \text{Undefined}$$

$$f\left(\frac{2\sqrt{2}}{3}\right) = \overset{3}{\cancel{9}}\left(\frac{2\sqrt{2}}{\cancel{3}}\right) + 8\left(\frac{2\sqrt{2}}{3}\right)^{-1} = 6\sqrt{2} + \overset{2\sqrt{2}}{\cancel{8}}\left(\frac{3}{\cancel{2\sqrt{2}}}\right) = 12\sqrt{2}$$

To check the sign of f' we will use -1 to represent $\left(-\infty, -\frac{2\sqrt{2}}{3}\right)$, -⅔ to represent $\left(-\frac{2\sqrt{2}}{3}, 0\right)$, ⅔ to represent $\left(0, \frac{2\sqrt{2}}{3}\right)$ and 1 to

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represent $\left(\frac{2\sqrt{2}}{3},\infty\right)$.

$$f'(-1)=\frac{9(-1)^2-8}{(-1)^2}=\frac{+}{+};f'\left(-\frac{2}{3}\right)=\frac{9\left(-\frac{2}{3}\right)^2-8}{\left(-\frac{2}{3}\right)^2}=\frac{\not{9}\left(\frac{4}{\not{9}}\right)-8}{+}=\frac{-}{+}$$

$$f'\left(\frac{2}{3}\right)=\frac{9\left(\frac{2}{3}\right)^2-8}{\left(\frac{2}{3}\right)^2}=\frac{\not{9}\left(\frac{4}{\not{9}}\right)-8}{+}=\frac{-}{+};f'(1)=\frac{9(1)^2-8}{(1)^2}=\frac{+}{+}$$

We then interpret the signs of f' as telling us the direction of f. That is, where f' is negative (-), f is decreasing (↘), and where f' is positive (+), f is increasing (↗).

From the table we can see that at $x=\frac{2\sqrt{2}}{3}$, f has a local maximum, the value being $-12\sqrt{2}$, and at $x=\frac{2\sqrt{2}}{3}$, f has a local minimum, the value being $12\sqrt{2}$.

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8. Consider the function $f(x = 12x^5 + 75x^4 - 120x^3 + 1$. For this function there are four important intervals: $(-\infty,A]$, $[A,B]$, $[B,C]$, and $[C,\infty)$, where A, B, and C are critical numbers.
   Determine A: **-6**, B: **0**, and C: **1**
   At each critical number, A, B and C, determine whether f has a local minimum, a local maximum, or neither.
   A: min. **MAX.** neither B: min. MAX. **neither** C: **min.** MAX. neither

Solution:
Step one is to determine f'.

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(12x^5 + 75x^4 - 120x^3 + 1\right)}{dx}$$
$$= \frac{d12x^5}{dx} + \frac{d75x^4}{dx} - \frac{d120x^3}{dx} + \frac{d1}{dx} \quad \text{(Sum \& Difference Rules)}$$
$$= 12\frac{dx^5}{dx} + 75\frac{dx^4}{dx} - 120\frac{dx^3}{dx} + 0 \quad \text{(Constant Multiple \& Constant Rules)}$$
$$= 12\left(5x^4\right) + 75(4x^3) - 120(3x^2) \quad \text{(Power \& Identity Rules)}$$
$$= 60x^4 + 300x^3 - 360x^2 \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$= 60x^2\left(x^2 + 5x - 6\right) \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$
$$= 60x^2\left(x+6\right)\left(x-1\right) \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

Step two is to determine all critical numbers (values of x that make f' be 0 or discontinuous)
Since f' is a polynomial and continuous everywhere, there are no critical numbers due to f' being discontinuous.

$$f' = 0 \Leftrightarrow 60x^2(x + 6)(x - 1) = 0 \Leftrightarrow x = -6, 0, 1$$

Step three is to create a miniature table recording the critical numbers, the value of f at these numbers (called critical values), and the sign of f' on intervals around the critical numbers

		M		SP		*m*	
x		-6		0		1	
f (*x*)	↗ ↗ ↗	→ 29809	↘ ↘ ↘	0 →	↘ ↘ ↘	-32 →	↗ ↗ ↗
f ′(*x*)	+ + +	0	- - -	0	- - -	0	+ + +

$$f(-6) = 12(-6)^5 + 75(-6)^4 - 120(-6)^3 + 1 = -93312 + 97200 + 25920 + 1 = 29809$$
$$f(0) = 12(0)^5 + 75(0)^4 - 120(0)^3 + 1 = 1$$
$$f(1) = 12(1)^5 + 75(1)^4 - 120(1)^3 + 1 = 12 + 75 - 120 + 1 = -32$$

To check the sign of f' we will use -10 to represent $(-\infty,-6)$, -1 to represent $(-6,0)$, ½ to represent $(0,1)$ and 2 to represent $(1,\infty)$.

$$f'(-10) = +\times+\times-\times- = +;\ f'(-1) = +\times+\times+\times- = -;\ f'(½) = +\times+\times+\times- = -;\ f'(2) = +\times+\times+\times+ = +$$

We then interpret the signs of f' as telling us the direction of f. That is, where f' is negative (-), f is decreasing (↘), and where f' is positive (+), f is increasing (↗).
From the table we can see that at $x = -6$, f has a local maximum, the value being 29809, at $x = 0$, f has a stationary point (neither a local maximum nor a local minimum, but rather a point on the graph where the function "hesitates"), and at $x = 1$, f has a local minimum, the value being -32.

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9. The function $f(x) = 2x^3 - 6x^2 - 90x - 1$ is decreasing on the interval (_______,_______).
   The function f is increasing on the intervals **(-∞,-3)** and **(5,∞)**.
   The function f has a local maximum at x = **-3**.

Solution:
Step one is to determine f'.

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(2x^3 - 6x^2 - 90x - 1\right)}{dx}$$
$$= \frac{d2x^3}{dx} - \frac{d6x^2}{dx} - \frac{d90x}{dx} - \frac{d1}{dx} \text{ (Sum \& Difference Rules)}$$
$$= 2\frac{dx^3}{dx} - 6\frac{dx^2}{dx} - 90\frac{dx}{dx} - 0 \text{ (Constant Multiple \& Constant Rules)}$$
$$= 2\left(3x^2\right) - 6(2x) - 90(1) \text{ (Power \& Identity Rules)}$$
$$= 6x^2 - 12x - 90 \text{ (Arithmetic/Algebra/Functions Cleanup)}$$
$$= 6\left(x^2 - 2x - 15\right) \text{ (Arithmetic/Algebra/Functions Cleanup)}$$
$$= 6(x+3)(x-5) \text{ (Arithmetic/Algebra/Functions Cleanup)}$$

Step two is to determine all critical numbers (values of x that make f' be 0 or discontinuous)
Since f' is a polynomial and continuous everywhere, there are no critical numbers due to f' being discontinuous.

$$f' = 0 \Leftrightarrow 6(x + 3)(x - 5) = 0 \Leftrightarrow x = -3, 5$$

Step three is to create a miniature table recording the critical numbers, the value of f at these numbers (called critical values), and the sign of f' on intervals around the critical numbers

		M		*m*	
x		-3		5	
$f(x)$	↗ ↗ ↗	→ 161	↘ ↘ ↘	-351 →	↗ ↗ ↗
$f'(x)$	+ + +	0	- - -	0	+ + +

$$f(-3) = 2(-3)^3 - 6(-3)^2 - 90(-3) - 1 = -54 - 54 + 270 - 1 = 161$$
$$f(5) = 2(5)^3 - 6(5)^2 - 90(5) - 1 = 250 - 150 - 450 - 1 = -351$$

To check the sign of f' we will use -4 to represent (-∞,-3), 0 to represent (-3,5) and 6 to represent (5,∞).

$$f'(-4) = +\times-\times- = +; f'(0) = +\times+\times- = -; f'(6) = +\times+\times+ = +$$

We then interpret the signs of f' as telling us the direction of f. That is, where f' is negative (-), f is decreasing (↘), and where f' is positive (+), f is increasing (↗).
From the table we can see that at $x = -3$, f has a local maximum, the value being 161, and at $x = 5$, f has a local minimum, the value being -351.

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10. Consider the function $f(x) = x^2e^{2x}$. For this function there are three important intervals: $(-\infty,A]$, $[A,B]$ and $[B,\infty)$, where A and B are critical numbers for f.
    Determine A: **-1** and B: **0**
    For each of the following intervals indicate whether f is Increasing or Decreasing.
    $(-\infty,A]$: **Inc.** ~~Dec.~~ $[A,B]$: ~~Inc.~~ **Dec.** $[B,\infty)$: **Inc.** ~~Dec.~~

Solution:
Step one is to determine f'.

$$f'(x) = \frac{df(x)}{dx} = \frac{d\left(x^2e^{(2x)}\right)}{dx} = \left(\frac{dx^2}{dx}\right)e^{(2x)} + x^2\left(\frac{de^{(2x)}}{dx}\right) \quad \textbf{(Product Rule)}$$

$$= (2x)e^{(2x)} + x^2\left(\frac{de^{(2x)}}{d(2x)} \times \frac{d2x}{dx}\right) \quad \textbf{(Power \& Chain Rule)}$$

$$= 2xe^{(2x)} + x^2\left(e^{(2x)} \times 2\frac{dx}{dx}\right) \quad \textbf{(e \& Constant Multiple Rules)}$$

$$= 2xe^{(2x)} + x^2\left(e^{(2x)} \times 2(1)\right) \quad \textbf{(Identity Rule)}$$

$$= 2xe^{(2x)} + 2x^2e^{(2x)} = 2x(1+x)e^{(2x)} \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

Step two is to determine all critical numbers (values of x that make f' be 0 or discontinuous)
Since f' is a product of a polynomial and an exponential growth function, both continuous everywhere, hence f' is also continuous everywhere and there are no critical numbers due to f' being discontinuous.
We also note that the exponential growth function factor is never 0 (think of the graph of exponential growth functions), hence, when examining where $f' = 0$, we can ignore the exponential factor.

$$f' = 0 \Leftrightarrow 2x(1 + x) = 0 \Leftrightarrow x = -1, 0$$

Step three is to create a miniature table recording the critical numbers, the value of f at these numbers (called critical values), and the sign of f' on intervals around the critical numbers

		M		*m*	
x		-1		0	
$f(x)$	↗ ↗ ↗	→ 0.1353	↘ ↘ ↘	0 →	↗ ↗ ↗
$f'(x)$	+ + +	0	- - -	0	+ + +

$$f(-1) = (-1)^2e^{(2(-1))} = e^{-2} = \frac{1}{e^2} \doteq 0.1353$$

$$f(0) = (0)^2e^{(2(0))} = 0$$

To check the sign of f' we will use -2 to represent $(-\infty,-1)$, -½ to represent $(-1,0)$ and 1 to represent $(0,\infty)$.
$2x(1 + x)e^{(2x)}$

$$f'(-2) = +\times-\times-\times+ = +;\ f'(-½) = +\times-\times+\times = -;\ f'(6) = +\times+\times+ = +$$

We then interpret the signs of f' as telling us the direction of f. That is, where f' is negative (-), f is decreasing (↘), and where f' is positive (+), f is increasing (↗).
From the table we can see that at $x = -1$, f has a local maximum, the value being approximately 0.1353, and at $x = 0$, f has a local minimum, the value being 0.

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11. Consider the function $f(x)=\dfrac{3x+4}{5x+3}$. For *f* there are two important intervals: $(-\infty,A)$ and (A,∞), where *f* is not defined at $x = A$.

Determine *A*: **-0.6**

For each of the following intervals indicate whether *f* is Increasing or Decreasing:

f on $(-\infty,A)$: ~~Inc.~~ **Dec.** *f* on (A,∞): ~~Inc.~~ **Dec.**

Note that this function has no inflection points, but we can still consider its concavity. For each of the following intervals indicate whether *f* is concave up or concave down.

f on $(-\infty,A)$: ~~Concave Up~~ **Concave Down** *f* on (A,∞): **Concave Up** ~~Concave Down~~

Solution:

We note that *f* is a rational function. Rational functions are continuous everywhere they are defined, which is everywhere that the denominator is not zero. Since the denominator here is a linear function we can easily see that *f* is not defined (and not continuous) where $5x + 3 = 0 \Leftrightarrow x = -3/5$. Thus, $A = -3/5 = -0.6$. To determine where any function is increasing or decreasing we examine the pattern of signs (+ or -) of f'. To determine where any function is concave up or down we examine the pattern of signs (+ or -) of f''. Thus, our next step is to compute and simplify both f' and f''.

$$f'(x)=\frac{df(x)}{dx}=\frac{d\left(\frac{3x+4}{5x+3}\right)}{dx}=\frac{\left(\frac{d(3x+4)}{dx}(5x+3)-(3x+4)\frac{d(5x+3)}{dx}\right)}{(5x+3)^2}\quad\text{(Quotient Rule)}$$

$$=\frac{\left(\left(\frac{d(3x)}{dx}+\frac{d4}{dx}\right)(5x+3)-(3x+4)\left(\frac{d(5x)}{dx}+\frac{d3}{dx}\right)\right)}{(5x+3)^2}\quad\text{(Sum Rule)}$$

$$=\frac{\left(\left(3\frac{dx}{dx}+(0)\right)(5x+3)-(3x+4)\left(5\frac{dx}{dx}+(0)\right)\right)}{(5x+3)^2}\quad\text{(Constant Multiple \& Constant Rule)}$$

$$=\frac{\left((3(1))(5x+3)-(3x+4)(5(1))\right)}{(5x+3)^2}\quad\text{(Identity Rule)}$$

$$=\frac{\left(3(5x+3)-(3x+4)5\right)}{(5x+3)^2}\quad\text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$=\frac{\left(\cancel{15x}+9-\cancel{15x}-20\right)}{(5x+3)^2}\quad\text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$=\frac{-11}{(5x+3)^2}=-11(5x+3)^{-2}\quad\text{(Arithmetic/Algebra/Functions Cleanup)}$$

$$f''(x)=\frac{d\left(\frac{df(x)}{dx}\right)}{dx}=\frac{d\left(-11(5x+3)^{-2}\right)}{dx}=-11\frac{d(5x+3)^{-2}}{dx}\quad\text{(Constant Multiple Rule)}$$

$$=-11\frac{d(5x+3)^{-2}}{d(5x+3)}\times\frac{d(5x+3)}{dx}\quad\text{(Chain Rule)}$$

$$=-11\left(-2(5x+3)^{-3}\right)\times\left(\frac{d5x}{dx}+\frac{d3}{dx}\right)\quad\text{(Power \& Sum Rules)}$$

$$=22(5x+3)^{-3}\times\left(5\frac{dx}{dx}+0\right)\quad\text{(Constant Multiple \& Constant Rules)}$$

$$=22(5x+3)^{-3}\times\left(5(1)\right)\quad\text{(Identity Rule)}$$

$$=110(5x+3)^{-3}=\frac{110}{(5x+3)^3}\quad\text{(Constant Multiple \& Constant Rule)}$$

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We can see that both f' and f'' have constant numerators, and denominators that are powers of the linear function, $(5x + 3)$. Thus, they can never be 0 (since their numerators clearly cannot be 0), and are both undefined (and hence discontinuous) at the same x value that f was undefined, namely $x = -3/5 = -0.6$.
Thus, to test the signs of both f' and f'' we can use -1 to represent the interval $(-\infty,-0.6)$ and 0 to represent the interval $(-0.6,\infty)$.

$$f'(-1) = -\times+ = -;\ f'(0) = -\times+ = -$$
$$f''(-1) = +\times- = -;\ f''(0) = +\times+ = +$$

Now, putting this information in a table we can easily see what must be happening.

			VA		
x		-0.6^-	-0.6	-0.6^+	
$f(x)$	↘ ↘ ↘	↘ ↓	U	↑ ↖	↘ ↘ ↘
$f'(x)$	- - -	$-\infty$	U	$+\infty$	- - -
$f''(x)$	- - -		U		+ + +
	⌢ ⌢ ⌢				⌣ ⌣ ⌣

Clearly f is both decreasing and concave down on $(-\infty,-0.6)$ and decreasing and concave up on $(-0.6,\infty)$.
Note that f has no local maxima or minima. Further, although the concavity of f changes as x moves past $x = -0.6$, that value of x is not called a point of inflection because in fact there is no point (f is not defined at $x = -0.6$). In fact, we can see that f has a vertical asymptote at $x = -0.6$

12. Consider the function $f(x) = 5(x - 4)^{2/3}$. This function has two important intervals:$(-\infty,A)$ and (A,∞), where f has a critical number at $x = A$.
    Determine A: **4**
    For each of the following intervals indicate whether f is Increasing or Decreasing:
    f on $(-\infty,A)$: ~~Inc.~~ **Dec.** f on (A,∞): **Inc.** ~~Dec.~~
    For each of the following intervals indicate whether f is concave up or concave down.
    f on $(-\infty,A)$: ~~Concave Up~~ **Concave Down** f on (A,∞): ~~Concave Up~~ **Concave Down**

Solution:
To complete this problem we need to know both f' and f'', so this is our first step.

$$f'(x) = \frac{df(x)}{dx} = \frac{d\,5(x-4)^{2/3}}{dx} = 5\frac{d\,(x-4)^{2/3}}{dx} \quad \text{(Constant Multiple Rule)}$$
$$= 5\frac{d\,(x-4)^{2/3}}{d\,(x-4)} \times \frac{d\,(x-4)}{dx} \quad \text{(Chain Rule)}$$
$$= 5\left(\tfrac{2}{3}\right)(x-4)^{-1/3} \times \left(\frac{dx}{dx} - \frac{d4}{dx}\right) \quad \text{(Power \& Difference Rules)}$$
$$= 5\left(\tfrac{2}{3}\right)(x-4)^{-1/3} \times (1-0) \quad \text{(Identity \& Constant Rules)}$$
$$= \left(\tfrac{10}{3}\right)(x-4)^{-1/3} = \frac{10}{3(x-4)^{1/3}} \quad \text{(Arithmetic/Algebra/Functions Cleanup)}$$

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$$f''(x) = \frac{d\left(\frac{df(x)}{dx}\right)}{dx} = \frac{d\left(\frac{10}{3}\right)(x-4)^{-1/3}}{dx} = \left(\frac{10}{3}\right)\frac{d(x-4)^{-1/3}}{dx} \quad \textbf{(Constant Multiple Rule)}$$

$$= \left(\frac{10}{3}\right)\frac{d(x-4)^{-1/3}}{d(x-4)} \times \frac{d(x-4)}{dx} \quad \textbf{(Chain Rule)}$$

$$= \left(\frac{10}{3}\right)\left(-\frac{1}{3}\right)(x-4)^{-4/3} \times \left(\frac{dx}{dx} - \frac{d4}{dx}\right) \quad \textbf{(Power \& Difference Rules)}$$

$$= \left(-\frac{10}{9}\right)(x-4)^{-4/3} \times (1-0) \quad \textbf{(Identity \& Constant Rules)}$$

$$= \left(-\frac{10}{9}\right)(x-4)^{-4/3} = \frac{-10}{9(x-4)^{4/3}} \quad \textbf{(Arithmetic/Algebra/Functions Cleanup)}$$

We can see that both f' and f'' have constant numerators, and denominators that are powers of the linear function, $(x - 4)$. Thus, they can never be 0 (since their numerators clearly cannot be 0), and are both undefined (and hence discontinuous) at the same x value, namely $x = 4$.
Thus, to test the signs of both f' and f'' we can use 0 to represent the interval $(-\infty,4)$ and 5 to represent the interval $(4,\infty)$.

$f'(0) = +/(+\times-) = -$; $f'(5) = +/(+\times+) = +$

$f''(0) = -/(+\times+) = -$; $f''(5) = -/(+\times+) = -$

Now, putting this information in a table we can easily see what must be happening.

		m	
x		4	
f (*x*)	↘ ↘ ↘	0	↗ ↗ ↗
f ′(*x*)	- - -	U	+ + +
f ″(*x*)	- - -	U	- - -
	⌢ ⌢ ⌢		⌢ ⌢ ⌢

Clearly f is both decreasing and concave down on $(-\infty,4)$ and increasing and concave down on $(4,\infty)$. Even though both f' and f'' are not defined at $x = 4$, we see that $f(4) = 0$. In fact, it would appear that f has what is called a "cusp" or "sharp point" at $x = 4$, and that 0 that occurs at $x = 4$ is a local minimum value for f.

13. Determine values for the constants a and b such that a graph of the function $f(x) = \frac{ax+7}{4-bx}$ will have a vertical asymptote at $x = -2$ and a horizontal asymptote $y = 3$.

a: **6** *b*: **-2**

Solution:
Given a rational function f, we can determine whether or not there is a vertical asymptote and where it is simply by determining where the denominator is zero, as long as the numerator is not also zero there. Of course if both are zero, we factor both the numerator and denominator, cancel the common factor, and try again. In this case both the numerator and denominator are linear functions so no factoring is necessary. We can see that the denominator is zero if and only if $4 - bx = 0 \Leftrightarrow x = 4/b$, assuming $b \neq 0$. Similarly the numerator is zero if and only if $ax + 7 = 0 \Leftrightarrow x = -7/a$, assuming $a \neq 0$. Now, since we are told that there should be a vertical asymptote at $x = -2$, this means that $4/b = -2 \Leftrightarrow b = -2$. We also need that $-7/a \neq -2 \Leftrightarrow a \neq 7/2$.
Now given a rational function f, we can determine whether or not there is a horizontal asymptote, and if there is, what horizontal line the horizontal asymptote is, by taking the limit of f as x approaches $\pm\infty$.

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$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{ax+7}{4-bx}\right) = \lim_{x\to\pm\infty}\left(\frac{a\cancel{x}}{-b\cancel{x}}\right)\lim_{x\to\pm\infty}\left(\frac{a}{-b}\right) = -\left(\frac{a}{b}\right)$$

Now, since we are told that the horizontal asymptote should be $y = 3$, this means that $3 = a/b$. Since from above we have $b = -2$, thus $3 = -(a/(-2)) \Leftrightarrow a = 6$, which we note is not equal to 7/2.

14. A graph of $f(x) = \dfrac{9x^2+9x+7}{3x^2-5x-5}$ has a horizontal asymptote $y = \mathbf{3}$.

 A graph of f crosses this horizontal asymptote at $x =$ **-11/12**.

Solution:

The given function f is a rational function so we can determine whether it has a horizontal asymptote and the precise horizontal line that is the asymptote by computing the limit of f as x approaches $\pm\infty$.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{9x^2+9x+7}{3x^2-5x-5}\right) = \lim_{x\to\pm\infty}\left(\frac{9\cancel{x^2}}{3\cancel{x^2}}\right) = \lim_{x\to\pm\infty}\left(\frac{9}{3}\right) = \frac{9}{3} = 3$$

Determining where f crosses this horizontal asymptote of $y = 9/3$, like solving for x-intercepts, consists of equating $f(x)$ to 3 (instead of 0), and solving for x.

$$\left(\frac{9x^2+9x+7}{3x^2-5x-5}\right) = 3 \Leftrightarrow \left(9x^2+9x+7\right) = 3\left(3x^2-5x-5\right)$$

$$\Leftrightarrow \cancel{9x^2} + 9x + 7 = \cancel{9x^2} - 15x - 15$$

$$\Leftrightarrow 9x + 15x = -7 - 15 \Leftrightarrow 24x = -22 \Leftrightarrow x = \frac{-22}{24} = \frac{-11}{12}$$

15. The Mayor of Saint-Louis-du-Ha!-Ha!, M. Donald Viel has hired a business consultant to plan for the future of his town. The business consultant estimates that the population is predicted by the function $P(t) = \dfrac{4080}{1+3.6e^{-0.25t}}$, which has

 $P'(t) = \dfrac{dP(t)}{dt} = \dfrac{3672e^{-0.25t}}{\left(1+3.6e^{-0.25t}\right)^2}$ and $P''(t) = \dfrac{d^2P(t)}{dt^2} = \left(\dfrac{918e^{-0.25t}}{\left(1+3.6e^{-0.25t}\right)^3}\right)\left(3.6e^{-0.25t}-1\right)$. The Mayor wants to know several things from the consultant, but the consultant is busy, so knowing that you are studying Calculus, His Honour comes to you instead.

 First, he wants to know what this model predicts for the long term population of
 Saint-Louis-du-Ha!-Ha!: **4080**

 Next, he wants to know whether in 20 years from now the rate of growth of the population will be
 increasing or decreasing: **decreasing**

 Finally, he wants to know when the rate of growth of the population will switch
 between increasing and decreasing: **5.12**

Solution:

The "long term" population would be determined by computing the limit of P as t approaches ∞.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left(\frac{4080}{1+3.6e^{-0.25t}}\right) = \left(\frac{4080}{1+3.6(0)}\right) = 4080$$

To determine if any function is increasing or decreasing we look at the sign of the function's derivative (a positive derivative means the function is increasing, a negative derivative means the function is decreasing). Now, the function expressing the rate of growth of the population is $P'(t)$, so to determine if $P'(t)$ is increasing or decreasing we should examine the sign of $P''(t)$:

$$P''(20) = \left.\frac{d^2P(t)}{dt^2}\right|_{t=20} = \left(\frac{918e^{-0.25(20)}}{\left(1+3.6e^{-0.25(20)}\right)^3}\right)\left(3.6e^{-0.25(20)}-1\right) \doteq -6.8$$

Thus, the rate of growth of the population is decreasing at $t = 20$.

For the rate of growth of the population to switch between increasing and decreasing, the derivative of the rate of growth of the

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population, namely $P''(t)$, has to either be 0 or discontinuous (as a consequence of the Intermediate Value Theorem).

$$P''(t) = \frac{d^2P(t)}{dt^2} = \left(\frac{918e^{-0.25t}}{\left(1+3.6e^{-0.25t}\right)^3}\right)\left(3.6e^{-0.25t}-1\right) = 0 \Leftrightarrow \left(3.6e^{-0.25t}-1\right) = 0$$

$$\Leftrightarrow 3.6e^{-0.25t} = 1 \Leftrightarrow \frac{1}{e^{0.25t}} = \frac{1}{3.6} \Leftrightarrow e^{0.25t} = 3.6 \Leftrightarrow \ln\left(e^{0.25t}\right) = \ln(3.6) \Leftrightarrow \frac{t}{4} = \ln(3.6)$$

$$\Leftrightarrow t = 4\ln(3.6) \doteq 5.12$$

We note that exponential functions are continuous everywhere. The two factors in the numerator of $P''(t)$ are both basically exponential functions, hence continuous everywhere. The denominator of $P''(t)$ is also an exponential function, so also continuous everywhere, but $P''(t)$ would be discontinuous at any value of t that made the denominator equal to 0. However, in this example that would mean that the exponential $3.6\ e^{-0.25t}$ would have to equal -1, and that is not possible. Thus, the only possible place for $P''(t)$ to change sign is t approximately equal to 5.12. That is, the rate of growth of the population can only change direction at t approximately equal to 5.12. We already know (from trying $t = 20$) that after $t = 5.12$ the function $P''(t)$ is negative. Testing the sign of $P''(t)$ for t less than $t = 5.12$, we compute $P''(t)$ at $t = 0$: $P''(0) = (918\ e^0)(3.6\ e^0 - 1)/(1 + 3.6\ e^0) = +\times+/+ = +$. Thus, prior to $t = 5.12$ the rate of growth of the population will be increasing.

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Paper Assignment #10 with Solutions

1. Consider the function $f(x) = 2x^2 - 8x + 3$, on the interval $0 \le x \le 8$. The global (absolute) maximum of $f(x)$ (on the given interval) is **67** and the global (absolute) minimum of $f(x)$ (on the given interval) is **-5**.

Solution:
We note that the given function f is a polynomial, hence continuous everywhere. Further, the interval given is a closed interval, [0,8]. This means our strategy for solving the problem is clear: 1) determine all critical numbers of f; 2) evaluate f at all critical numbers that lie within the given interval, as well as at the two end points of the interval, 0 and 8, and the smallest of these f values is the global (absolute) minimum while the largest is the global (absolute) maximum.
Step 1: Determine the critical numbers of f.
First we compute $f'(x)$: $f'(x) = 2(2x) - 8(1) + 0 = 4x - 8$.
Next we solve the equation, $f'(x) = 0$: $4x - 8 = 0 \Leftrightarrow 4x = 8 \Leftrightarrow x = 2$
Next we check where $f'(x)$ is discontinuous: since $f'(x)$ is a linear function, it is never discontinuous
Thus, the only critical number of f is $x = 2$, which happens to lie within the given interval.
Step 2: Evaluate f at $x = 0, 2, 8$.

$f(0) = 3$; $f(2) = 2(2)^2 - 8(2) + 3 = 8 - 16 + 3 = -5$; $f(8) = 2(8)^2 - 8(8) + 3 = 2(64) - 64 + 3 = 67$

Clearly the largest of these three f values is 67, so this is the global maximum, and this occurs at $x = 8$. Similarly, the smallest of these three f values is -5, so this is the global minimum, and this occurs at $x = 2$.
N.B. It is important to note that this question asks for the global maximum and minimum values, values of $f(x)$, not values of x. Thus, it is the $f(x)$ values that we report.

2. Consider the function $f(x) = x^4 - 72x + 2$, on the interval $-5 \le x \le 13$. The global (absolute) maximum of $f(x)$ (on the given interval) is **27,627**. and the global (absolute) minimum of $f(x)$ (on the given interval) is **-139.52**.

Solution:
We note that the given function f is a polynomial, hence continuous everywhere. Further, the interval given is a closed interval, [-5,13]. This means our strategy for solving the problem is clear: 1) determine all critical numbers of f; 2) evaluate f at all critical numbers that lie within the given interval, as well as at the two end points of the interval, -5 and 13, and the smallest of these f values is the global (absolute) minimum while the largest is the global (absolute) maximum.
Step 1: Determine the critical numbers of f.
First we compute $f'(x)$: $f'(x) = 4x^3 - 72(1) + 0 = 4(x^3 - 18)$.
Next we solve the equation, $f'(x) = 0$: $4(x^3 - 18) = 0 \Leftrightarrow x^3 = 18 \Leftrightarrow x = (18)^{1/3}$
Next we check where $f'(x)$ is discontinuous: since $f'(x)$ is a polynomial, it is never discontinuous
Thus, the only critical number of f is $x = (18)^{1/3}$, which happens to lie within the given interval.
Step 2: Evaluate f at $x = -5, (18)^{1/3}, 13$.
$f(-5) = 625 - 72(-5) + 2 = 987$; $f((18)^{1/3}) = ((18)^{1/3})^4 - 72((18)^{1/3}) + 2 \doteq -139.52$; $f(13) = (13)^4 - 72(13) + 2 = 27,627$
Clearly the largest of these three f values is 27,627, so this is the global maximum, and this occurs at $x = 13$. Similarly, the smallest of these three f values is -139.52, so this is the global minimum, and this occurs at $x = (18)^{1/3}$.
N.B. It is important to note that this question asks for the global maximum and minimum values, values of $f(x)$, not values of x. Thus, it is the $f(x)$ values that we report.

3. Consider the function $f(x) = 3x^{2/3} - 2x$, on the interval [-1,1]. The global (absolute) maximum of $f(x)$ (on the given interval) occurs at $x =$ **-1** and the global (absolute) minimum of $f(x)$ (on the given interval) occurs at $x =$ **0**

Solution:
We note that the given function f is a difference of a power function with positive exponent, $3x^{2/3}$, and a linear function, $2x$, both of which are continuous everywhere. Thus f is continuous everywhere as well. Further, the interval given is a closed interval, [-1,1]. This means our strategy for solving the problem is clear: 1) determine all critical numbers of f; 2) evaluate f at all critical numbers that lie within the given interval, as well as at the two end points of the interval, -1 and 1, and the smallest of these f values is the global (absolute) minimum while the largest is the global (absolute) maximum.
Step 1: Determine the critical numbers of f.

First we compute $f'(x)$: $f'(x) = 3\left(\frac{2}{3}\right)x^{-1/3} - 2(1) = \frac{2}{x^{1/3}} - 2\frac{x^{1/3}}{x^{1/3}} = \frac{2\left(1 - x^{1/3}\right)}{x^{1/3}}$.

Next we solve the equation, $f'(x) = 0$: $\frac{2\left(1 - x^{1/3}\right)}{x^{1/3}} = 0 \Leftrightarrow \left(1 - x^{1/3}\right) = 0 \Leftrightarrow x^{1/3} = 1 \Leftrightarrow x = 1$

Next we check where $f'(x)$ is discontinuous: $x^{1/3} = 0 \Leftrightarrow x = 0$

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Although we solved for two critical numbers, one of them, $x = 1$, is actually the same as an end point of the given interval, so we have really added only one new x-value of interest, $x = 0$, which happens to lie within the given interval.
Step 2: Evaluate f at $x = -1, 0, 1$.
$f(-1) = 3(-1)^{2/3} - 2(-1) = 5$; $f(0) = 3(0)^{2/3} - 2(0) = 0$; $f(1) = 3(1)^{2/3} - 2(1) = 1$
Clearly the largest of these three f values is 5, so this is the global maximum, and this occurs at $x = -1$. Similarly, the smallest of these three f values is 0, so this is the global minimum, and this occurs at $x = 0$.
N.B. It is important to note that this question asks for the *x*-values at which the global maximum and minimum values, values of *f* (*x*), are located. Thus, it is the *x*-values that we report.

4. Consider the function $f(t) = t\sqrt{4-t}$, on the interval [-1,3]. The global (absolute) maximum of $f(x)$ (on the given interval) occurs at $t =$ **2⅔** and the global (absolute) minimum of $f(x)$ (on the given interval) occurs at $t =$ **-1**

Solution:
We note that the given function f is a difference of a power function with positive exponent, $3x^{2/3}$, and a linear function, $2x$, both of which are continuous everywhere. Thus f is continuous everywhere as well. Further, the interval given is a closed interval, [-1,3]. This means our strategy for solving the problem is clear: 1) determine all critical numbers of f; 2) evaluate f at all critical numbers that lie within the given interval, as well as at the two end points of the interval, -1 and 3, and the smallest of these f values is the global (absolute) minimum while the largest is the global (absolute) maximum.
Step 1: Determine the critical numbers of f.
First we compute $f'(t)$:

$$f(t) = t\sqrt{4-t} = t(4-t)^{1/2}$$

$$f'(t) = \frac{df}{dt} = \frac{dt(4-t)^{1/2}}{dt} = \left(\frac{dt}{dt}\right)(4-t)^{1/2} + t\left(\frac{d(4-t)^{1/2}}{dt}\right)$$

$$= (1)(4-t)^{1/2} + t\left(\frac{d(4-t)^{1/2}}{d(4-t)} \times \frac{d(4-t)}{dt}\right) = (4-t)^{1/2} + t\left(\tfrac{1}{2}(4-t)^{-1/2} \times (0-1)\right)$$

$$= (4-t)^{1/2} \times \frac{2(4-t)^{1/2}}{2(4-t)^{1/2}} - \frac{t}{2(4-t)^{1/2}} = \frac{2(4-t)-t}{2(4-t)^{1/2}} = \frac{8-3t}{2\sqrt{4-t}}$$

Next we solve the equation, $f'(t) = 0$: $8 - 3t = 0 \Leftrightarrow t = 8/3 = 2⅔$
Next we check where $f'(t)$ is discontinuous: $2\sqrt{4-t} = 0 \Leftrightarrow 4 - t = 0 \Leftrightarrow t = 4$
Although we solved for two critical numbers, one of them, $t = 4$, is actually outside the given interval, so we have really added only one new t-value of interest, $t = 2⅔$, which happens to lie within the given interval.
Step 2: Evaluate f at $t = -1$, 2⅔, 3.

$$f(-1) = (-1)\sqrt{4-(-1)} = -\sqrt{5}$$

$$f\left(\frac{8}{3}\right) = \left(\frac{8}{3}\right)\sqrt{4-\left(\frac{8}{3}\right)} = \left(\frac{8}{3}\right)\sqrt{\frac{4}{3}} = \left(\frac{16}{3\sqrt{3}}\right) \doteq 3.08$$

$$f(3) = (3)\sqrt{4-(3)} = 3\sqrt{1} = 3$$

Clearly the largest of these three f values is 3.08, so this is the global maximum, and this occurs at $t = 2⅔$. Similarly, the smallest of these three f values is $-\sqrt{5}$, so this is the global minimum, and this occurs at $t = -1$.
N.B. It is important to note that this question asks for the *t*-values at which the global maximum and minimum values, values of *f* (*t*), are located. Thus, it is the *t*-values that we report.

5. Consider the function $g(x) = \dfrac{4x}{x^2+1}$, on the interval [-4,0]. The global (absolute) maximum of $g(x)$ (on the given interval) occurs at $x =$ **0** and the global (absolute) minimum of $g(x)$ (on the given interval) occurs at $x =$ **-1**

Solution:
We note that the given function g is a ratio or quotient of a linear functoin, $4x$, and a quadratic function, $x^2 + 1$, both of which are continuous everywhere. Thus g is continuous everywhere, as long as the denominator is not 0. Since $x^2 + 1 = 0 \Leftrightarrow x^2 = -1$, for which there is no real number solution, this means g is continuous everywhere. Further, the interval given is a closed interval, [-4,0]. This

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means our strategy for solving the problem is clear: 1) determine all critical numbers of g ; 2) evaluate g at all critical numbers that lie within the given interval, as well as at the two end points of the interval, -4 and 0, and the smallest of these g values is the global (absolute) minimum while the largest is the global (absolute) maximum.
Step 1: Determine the critical numbers of g.
First we compute $g'(x)$:

$$g'(x) = \frac{dg}{dx} = \frac{d\,\dfrac{4x}{x^2+1}}{dx} = \frac{\left(\dfrac{d4x}{dx}\right)\left(x^2+1\right) - 4x\left(\dfrac{d\left(x^2+1\right)}{dx}\right)}{\left(x^2+1\right)^2}$$

$$= \frac{(4)\left(x^2+1\right) - 4x(2x+0)}{\left(x^2+1\right)^2} = \frac{4x^2+4-8x^2}{\left(x^2+1\right)^2} = \frac{4-4x^2}{\left(x^2+1\right)^2} = \frac{4\left(1-x^2\right)}{\left(x^2+1\right)^2}$$

Next we solve the equation, $g'(x) = 0$: $4(1 - x^2) = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$
Next we check where $g'(x)$ is discontinuous: $(x^2 + 1)^2 = 0 \Leftrightarrow$ never, as discussed above
Although we solved for two critical numbers, one of them, $x = 1$, is actually outside of the given interval, so we have really added only one new x-value of interest, $x = -1$, which happens to lie within the given interval.
Step 2: Evaluate g at $x = -4, -1, 0$.

$$g(-4) = \frac{4(-4)}{(-4)^2+1} = -\frac{16}{17}; g(-1) = \frac{4(-1)}{(-1)^2+1} = -\frac{4}{2} = -2; g(0) = \frac{4(0)}{(0)^2+1} = \frac{0}{1} = 0$$

Clearly the largest of these three g values is 0, so this is the global maximum, and this occurs at $x = 0$. Similarly, the smallest of these three g values is -2, so this is the global minimum, and this occurs at $x = -1$.
N.B. It is important to note that this question asks for the *x*-values at which the global maximum and minimum values, values of *g* (*x*), are located. Thus, it is the *x*-values that we report.

6. One of the formulae for inventory management says that the average weekly cost of ordering paying for, and holding merchandise is $A(q) = \frac{km}{q} + cm + \frac{hq}{2}$, where q is the quantity that you order when items run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you place an order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Your job as inventory manager for your store is to determine the quantity q that will minimize $A(q)$, the average weekly cost, when $k = 1$, $m = 500$, $c = 5$, $h = 0.1$. (The formula determined with letters rather than numbers is called the *Wilson lot size formula*). $q =$ **1000**

Solution:
Once we replace the constants k, m, c and h with the given values we obtain a simpler formula for A:

$$A(q) = \frac{500}{q} + 2{,}500 + \frac{q}{20} = \frac{10{,}000 + 50{,}000q + q^2}{20q}$$

Since q is the quantity that one orders, it is apparent that $0 < q$, but there does not seem to be any upper limit on the value of q. That is, the interval for q is $(0,\infty)$, which is not closed. Since we do not have a continuous function on a closed interval our strategy for determining the quantity that will minimize A is essentially to gather enough information to "see" what a graph of A would look like. This consists of determining all critical numbers for A, and the sign of A' in the intervals surrounding the critical numbers, but within the interval $(0,\infty)$. That information would allow us to determine whether A has a global minimum, and if so, what value of q yields that global minimum.
Our first step is to compute A':

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$$A'(q)=\frac{dA}{dq}=\frac{d\left(\dfrac{10{,}000+50{,}000q+q^2}{20q}\right)}{dq}$$

$$=\frac{\left(\dfrac{d\left(10{,}000+50{,}000q+q^2\right)}{dq}\right)(20q)-\left(10{,}000+50{,}000q+q^2\right)\left(\dfrac{d20q}{dq}\right)}{(20q)^2}$$

$$=\frac{\left(0+50{,}000+2q\right)(20q)-\left(10{,}000+50{,}000q+q^2\right)\left(20(1)\right)}{20^2q^2}$$

$$=\cancel{20}\left[\frac{\cancel{50{,}000q}+2q^2-10{,}000-\cancel{50{,}000q}-q^2}{20^{\cancel{2}}q^2}\right]=\frac{q^2-10{,}000}{20q^2}$$

Next we solve for critical numbers:
$A' = 0$: $q^2 - 10{,}000 = 0 \Leftrightarrow q^2 = 10{,}000 \Leftrightarrow q = \pm 100$
A' is discontinuous $\Leftrightarrow 20q^2 = 0 \Leftrightarrow q = 0$
We note that the function A is in fact undefined at $q = 0$, and that $q = -100$ does not make any sense (is outside of our interval), thus we really only have one critical number $q = 100$.
Thus, there are two intervals (0,100) and (100,∞) of interest. We shall use $q = 1$ to represent the first of these, and $q = 200$ to represent the second of these. (These values were chosen to make computation simple.) Note that we are only interested in the sign of A', not the actual value.

$$A'(1)=\frac{(1)^2-10{,}000}{20(1)^2}=\frac{-}{+}=-;\ A'(200)=\frac{(200)^2-10{,}000}{20(200)^2}=\frac{+}{+}=+$$

			min		
q	0		100		∞
$A(q)$	U	↘ ↘ ↘	→	↗ ↗ ↗	
$A'(q)$	U	- - - - -	0	+ + + + +	

We can now easily see that A has a global minimum at $q = 100$.

7. An apple orchard produces annual revenue of \$50 per tree planted when planted with 1000 trees or less. Because of overcrowding, the annual revenue per tree is reduced by 2¢ per additional tree planted above 1000 trees. If the cost of maintaining each tree is \$10 per year, how many trees should
be planted to maximize total profit from the orchard? **1500**

Solution:
The phrase "maximize total profit" makes it clear that we need to construct a total profit function, where it would seem that the independent variable, x, would be the number of trees. In general, total profit is the difference between total revenue and total cost, so we begin by constructing functions for those.
average revenue, or revenue per tree, is \$50 - \$0.02×(x - 1,000). Thus total revenue, which is revenue per tree × # of trees, would be $(50 - 0.02(x - 1000))x$. We are given average cost, or cost per tree, as being \$10, so total cost, which is cost per tree × # of trees, would be $10x$.
Thus, total profit is $P(x) = (50 - 0.02(x - 1000))x - 10x = 50x - 0.02x^2 + 20x - 10x = 60x - 0.02x^2$.
It should be clear that $1000 \le x$, but there does not seem to be an upper limit for the number of trees. Hence the interval we are interested in is [1000,∞), which is not a closed interval. Thus, we will: 1) compute $P'(x)$; 2) determine all critical numbers of $P(x)$; 3) determine the sign of $P'(x)$ on the intervals surrounding the critical numbers, but within the interval of interest; 4) deduce whether $P(x)$ actually has a global maximum, and if so, at what value of x that maximum would occur.
1) $P'(x) = 60 - 0.02(2x) = 60 - 0.04x$
2) $P'(x) = 0 \Leftrightarrow 60 - 0.04x = 0 \Leftrightarrow x = 1500$; $P'(x)$ is never discontinuous since it is a linear function

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3) We use $x = 1100$ to represent the interval [1000,1500) and 2000 to represent the interval (1500,∞).

$$P'(1100) = 60 - 0.04(1100) = 60 - 44 = +;\ P'(2000) = 60 - 0.04(2000) = 60 - 80 = -$$

4) We make a small table to see what is happening.

			Max		
x	1000		1500		∞
$P(x)$		↗ ↗ ↗	→	↘ ↘ ↘	
$P'(x)$		+ + + + +	0	- - - - -	

We can now easily see that P has a global maximum at $x = 1500$.

8. You operate a tour service that offers the following rates:
   a) $200 per person if 50 (the minimum number allowed to book the tour) people go on the tour.
   b) For each additional person, up to a maximum of 80 people total, everyone's charge is reduced by $2.

It costs you $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit? # of people: **67**

Solutions:

The phrase "maximize your profit" clearly indicates that this is an optimization problem and that the function we are to "optimize" is the profit function. Thus, we must use the information given to contruct a profit or rather a total profit function. Since total profit is the difference between total revenue and total cost, we first construct each of these. Total revenue is revenue per person times the number of people. Let x represent the number of people. Then revenue per person is \$200 - \$2×(x - 50). Thus, the total revenue is $(200 - 2(x - 50))x$. Total cost is made up of fixed cost plus cost per person times the number of people, which here would be $6000 + 32x$. Finally, total profit is:

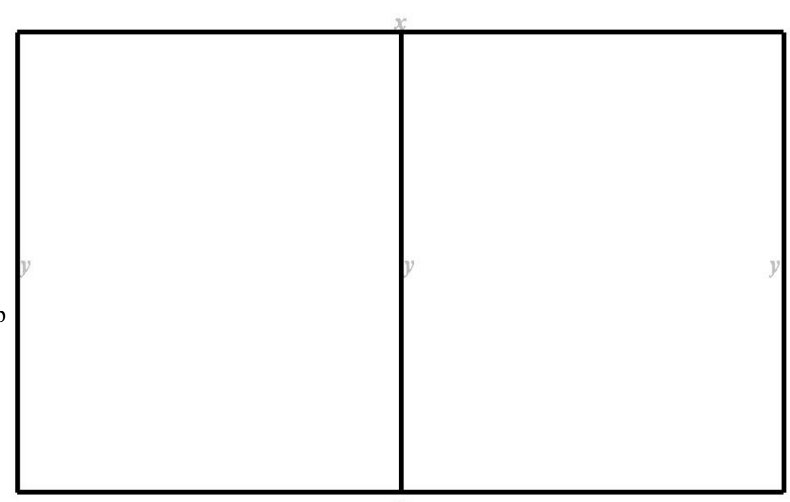
$$P(x) = (200 - 2(x - 50))x - (6000 + 32x) = 200x - 2x^2 + 100x - 6000 - 32x = -2x^2 + 268x - 6000$$

From the problem we note that $x \in [50,80]$. Since P is a quadratic function, which is continuous everywhere, and the interval of concern is closed, we have a simple two-step procedure: 1) determine all critical numbers of P in the given interval; 2) compute the value of P at all the above critical numbers, as well as at the two end points of the given interval, 50 and 80, and decide which is the maximum.

1) We note that $P'(x) = -2(2x) + 268(1) - 0 = -4x + 268$, so $P'(x) = 0 \Leftrightarrow -4x + 268 = 0 \Leftrightarrow x = 67$. We also note that $P'(x)$ is a linear function, hence continuous everywhere, so there are no other critical numbers. Finally, we note that 67 is inside the given interval [50,80].

2) $P(50) = -2(50)^2 + 268(50) - 6000 = -5,000 + 13,400 - 6,000 = 2,400$
   $P(67) = -2(67)^2 + 268(67) - 6000 = -8,978 + 17,956 - 6,000 = 2,978$
   $P(80) = -2(80)^2 + 268(80) - 6000 = -12,800 + 21,440 - 6,000 = 2,640$

From which we conclude that the maximum profit is $2,978, achieved when 67 passengers book the tour.



9. A rancher wants to fence in an area of 2,000,000 square feet in a rectangular field and then divide it in half with a fence down the middle, parallel to one side. What is the shortest length of fence that the rancher can use? **6928.2**

Solution:

The phrase "shortest length of fence" indicates that this is an optimization problem, and that the function we are to "optimize" is length of fence. We use a small diagram to help us visualize the length of fence.

We can see that if F is the length of the fence, then $F = 2x + 3y$. However, in Calculus I we can only work on optimization problems where the function is a function of a single independent variable. Thus, there must be more information given in the problem that will allow us to eliminate one of x and y. The phrase "area of 2,000,000 square feet in a rectangular field" tells us that $2{,}000{,}000 = xy \Leftrightarrow y = 2{,}000{,}000/x$. Substituting this into the expression for F we obtain: $F(x) = 2x + 3(2{,}000{,}000/x)$.

Using algebra we can rewrite this as: $F(x) = 2x + 3\left(\frac{2{,}000{,}000}{x}\right) = \frac{2x^2 + 6{,}000{,}000}{x}$

The best we can do in terms of an interval that we are interested in is to state that $0 < x$, or $x \in (0,\infty)$.
Although our function F is continuous on this interval (its only discontinuity, as a rational function, is where the denominator is 0, which is at $x = 0$), the interval is not closed. Thus our strategy involves: 1) determine all critical values in the interval we are concerned with; 2) determine the sign of F' on the intervals surrounding the critical values, but inside the interval of concern; 3) deduce whether F actually has a minimum value, and if it does, at what value of x the minimum occurs.

$$F'(x) = \frac{dF}{dx} = \frac{d\left(\frac{2x^2+6{,}000{,}000}{x}\right)}{dx} = \frac{\left(\frac{d\left(2x^2+6{,}000{,}000\right)}{dx}\right)x - \left(2x^2+6{,}000{,}000\right)\left(\frac{dx}{dx}\right)}{x^2}$$

$$= \frac{(2(2x))x - \left(2x^2+6{,}000{,}000\right)(1)}{x^2} = \frac{4x^2 - 2x^2 - 6{,}000{,}000}{x^2} = \frac{2x^2 - 6{,}000{,}000}{x^2}$$

1) $F'(x) = 0 \Leftrightarrow 2x^2 - 6{,}000{,}000 = 0 \Leftrightarrow x^2 = 3{,}000{,}000 \Leftrightarrow x = \pm 1000\sqrt{3},$

but $-1000\sqrt{3}$ is outside the interval we are concerned with

$F'(x)$ is discontinuous $\Leftrightarrow x = 0$, which is outside the interval we are concerned with

2) We see that there are two intervals, $\left(0, 1000\sqrt{3}\right)$ **and** $\left(1000\sqrt{3}, \infty\right)$, and we will use 1000 to represent the first interval, 2000 to represent the second interval:

$$F'(1000) = \frac{2(1000)^2 - 6{,}000{,}000}{(1000)^2} = \frac{2{,}000{,}000 - 6{,}000{,}000}{(1000)^2} = \frac{-}{+} = -$$

$$F'(2000) = \frac{2(2000)^2 - 6{,}000{,}000}{(2000)^2} = \frac{8{,}000{,}000 - 6{,}000{,}000}{(2000)^2} = \frac{+}{+} = +$$

3) We record the information gathered in a table and can easily see that F has a global minimum value for this interval at $x = 1000\sqrt{3}$, and the global minimum is:

$$F\left(1000\sqrt{3}\right) = \frac{2\left(1000\sqrt{3}\right)^2 + 6{,}000{,}000}{\left(1000\sqrt{3}\right)} = \frac{6{,}000{,}000 + 6{,}000{,}000}{1000\sqrt{3}}$$

$$= \frac{12{,}000{,}000}{1000\sqrt{3}} = 4{,}000\sqrt{3} \doteq 6928.2$$

			min		
x	0		$1000\sqrt{3}$		∞
$F(x)$		↘ ↘ ↘	6928.2 →	↗ ↗ ↗	
$F'(x)$		- - - - -	0	+ + + + +	

10. Given the cost function $C(x) = 250\sqrt{x} + \frac{x^2}{27{,}000}$, where x is the level of production, determine the following information:

Appendix E - Common Assignment Problems

a. The cost at a production level of 1,300: **9076.47**
b. The average cost at a production level of 1,300: **6.98**
c. The marginal cost at a production level of 1,300: **3.56**
d. The production level that will minimize the average cost: **22,500**
e. The minimal average cost: **2.5**

Solution:

a. $$C(1300) = 250\sqrt{1300} + \frac{(1300)^2}{27{,}000} \doteq 9076.47$$

b. Average cost is cost/production level so

$$AC(x) = \frac{C(x)}{x} = \frac{250\sqrt{x} + \frac{x^2}{27{,}000}}{x}; AC(1{,}300) = \frac{250\sqrt{(1300)} + \frac{(1300)^2}{27{,}000}}{(1300)} \doteq 6.98$$

c. Marginal cost is just the derivative of cost so

$$C(x) = 250\sqrt{x} + \frac{x^2}{27{,}000} = 250x^{1/2} + \frac{1}{27{,}000}x^2$$

$$C'(x) = \frac{dC}{dx} = \frac{d\left(250x^{1/2} + \frac{1}{27{,}000}x^2\right)}{dx} = 250\left(\tfrac{1}{2}x^{-1/2}\right) + \frac{1}{27{,}000}(2x)$$

$$= \frac{125}{\sqrt{x}} + \frac{x}{13{,}500} = \frac{125 \times 13{,}500 + x\sqrt{x}}{13{,}500\sqrt{x}} = \frac{1{,}687{,}500 + x\sqrt{x}}{13{,}500\sqrt{x}}$$

$$C'(1300) = \frac{1{,}687{,}500 + (1300)\sqrt{(1300)}}{13{,}500\sqrt{(1300)}} \doteq 3.56$$

d. & e. The interval would be $(0,\infty)$ and

$$AC'(x) = \frac{dAC}{dx} = \frac{d\left(\frac{250\sqrt{x} + \frac{x^2}{27{,}000}}{x}\right)}{dx} = \frac{\left(\frac{d\left(250x^{1/2} + \frac{x^2}{27{,}000}\right)}{dx}\right)x - \left(250x^{1/2} + \frac{x^2}{27{,}000}\right)\left(\frac{dx}{dx}\right)}{x^2}$$

$$= \frac{\left(250\left(\tfrac{1}{2}\right)x^{-1/2} + \frac{2x}{27{,}000}\right)x - \left(250x^{1/2} + \frac{x^2}{27{,}000}\right)(1)}{x^2} = \frac{125x^{1/2} + \frac{2x^2}{27{,}000} - 250x^{1/2} - \frac{x^2}{27{,}000}}{x^2}$$

$$= \frac{-125x^{1/2} + \frac{x^2}{27{,}000}}{x^2} = \frac{-3{,}375{,}000x^{1/2} + x^2}{27{,}000x^2} = \cancel{x^{1/2}}\left(\frac{-3{,}375{,}000 + x^{3/2}}{27{,}000x^{\frac{3}{2}}}\right) = \frac{-3{,}375{,}000 + x^{3/2}}{27{,}000x^{3/2}}$$

Thus, we now solve for all critical numbers of average cost:

$$AC'(x) = 0 \Leftrightarrow \frac{-3{,}375{,}000 + x^{3/2}}{27{,}000x^{3/2}} = 0 \Leftrightarrow -3{,}375{,}000 + x^{3/2} = 0 \Leftrightarrow x^{3/2} = 3{,}375{,}000 \Leftrightarrow x = 22{,}500$$

$$AC'(x) \textbf{ is discontinuous} \Leftrightarrow x^{3/2} = 0 \Leftrightarrow x = 0 \textbf{ but } x = 0 \textbf{ does not lie in the interval}$$

Thus there are two intervals on which we wish to know the sign of $AC'(x)$, (0,22,500) and (22,500,∞). We will use 1 to represent the first interval and 40,000 to represent the second interval.

$$AC'(1) = \frac{-3{,}375{,}000 + (1)^{3/2}}{27{,}000(1)^{3/2}} = \frac{-}{+} = -; AC'(40{,}000) = \frac{-3{,}375{,}000 + (40{,}000)^{3/2}}{27{,}000(40{,}000)^{3/2}} = \frac{+}{+} = +$$

Appendix E - Common Assignment Problems

$$AC(x)=\frac{250\sqrt{(22,500)}+\frac{(22,500)^2}{27,000}}{(22,500)}=\frac{5}{2}=2.5$$

			min		
x	0		22,500		∞
$AC(x)$		↘ ↘ ↘	2.5 →	↗ ↗ ↗	
$AC'(x)$		- - - - -	0	+ + + + +	

We can see that average cost has a global minimum of \$2.5 per unit produced when 22,500 units are produced.

11. Given the cost function $C(x) = 62,500 + 800x + x^2$, where x is the level of production, determine the following information:
    a. The cost at a production level of 1,400: **3,142,500**
    b. The average cost at a production level of 1,400: **2,244.64**
    c. The marginal cost at a production level of 1,400: **3,600**
    d. The production level that will minimize the average cost: **250**
    e. The minimal average cost: **1,300**

Solution:

a. $C(1,400) = 62,500 + 800(1,400) + (1,400)^2 = 3,142,500$

b. Average cost is just cost divided by level of production, so

$$AC(x)=\frac{62,500+800x+x^2}{x}\Rightarrow AC(1,400)=\frac{62,500+800(1,400)+(1,400)^2}{(1,400)}\doteq 2244.64$$

c. Marginal cost just means the derivative of cost, so

$$C'(x) = 0 + 800(1) + 2x \Rightarrow C'(1,400) = 800 + 2(1,400) = 3,600$$

d. & e.

The interval would be $(0,\infty)$ and

$$AC'(x)=\frac{dAC}{dx}=\frac{d\left(\frac{62,500+800x+x^2}{x}\right)}{dx}$$

$$=\frac{\left(\frac{d\left(62,500+800x+x^2\right)}{dx}\right)x-\left(62,500+800x+x^2\right)\left(\frac{dx}{dx}\right)}{x^2}$$

$$=\frac{\left(800(1)+2x\right)x-\left(62,500+800x+x^2\right)(1)}{x^2}=\frac{\cancel{800x}+\cancel{2}x^2-62,500-\cancel{800x}-\cancel{x^2}}{x^2}$$

$$=\frac{x^2-62,500}{x^2}$$

Thus, we now solve for all critical numbers of average cost:

$AC'(x) = 0 \Leftrightarrow x^2 - 62,500 = 0 \Leftrightarrow x = \pm 250$. We note that $x = -250$ is outside of the interval of interest.

$AC'(x)$ is discontinuous $\Leftrightarrow x^2 = 0 \Leftrightarrow x = 0$, which is outside of the interval of interest.

Thus there are two intervals on which we wish to know the sign of $AC'(x)$, $(0,250)$ and $(250,\infty)$. We will use 1 to represent the first interval and 400 to represent the second interval.

$$AC'(1)=\frac{(1)^2-62,500}{(1)^2}=\frac{-}{+}=-;AC'(400)=\frac{(400)^2-62,500}{(400)^2}=\frac{+}{+}=+$$

Appendix E - Common Assignment Problems

$$AC(250) = \frac{62{,}500 + 800(250) + (250)^2}{(250)} = 1{,}300$$

			min		
x	0		250		∞
$AC(x)$		↘ ↘ ↘	1,300 →	↗ ↗ ↗	
$AC'(x)$		- - - - -	0	+ + + + +	

We can see that average cost has a global minimum of $2.5 per unit produced when 22,500 units are produced.

12. Given the cost function $C(x) = 4750 + 360x + 1.3x^2$, where x is the level of production, and the demand function $p(x) = 1080 - 0.1x$, where p is the price, determine the level of production that will maximize profit: **257.14**

Solution:

Total profit is the difference between total revenue and total cost. Total revenue is the product of price per unit times the number of units produced, and here would be $p(x)\times x = 1080x - 0.1x^2$. Total cost has been given already (a sum of fixed cost, 4750, and cost per unit times the number of units produced, $(360 + 1.3x)\times x$. Thus, $TP(x) = (1080x - 0.1x^2) - (4750 + 360x + 1.3x^2) = -4750 + 720x - 1.4x^2$. The interval that we would be interested in here would be $(0,\infty)$. Thus, although the total profit function, a quadratic function, is continuous everywhere, the interval is not closed. Thus our strategy involves: 1) determine all critical values in the interval we are concerned with; 2) determine the sign of TP' on the intervals surrounding the critical values, but inside the interval of concern; 3) deduce whether TP actually has a maximum value, and if it does, at what value of x the maximum occurs.

1) $TP'(x) = 0 + 720(1) - 1.4(2x) = 720 - 2.8x \Rightarrow (TP'(x) = 0) \Leftrightarrow (720 - 2.8x = 0) \Leftrightarrow (x = 257.14)$
2) There are two intervals of interest: (0,257.14) and (257.14,∞). We will use 1 to represent the first interval, and 300 to represent the second interval.

$$TP'(1) = 720 - 2.8(1) = +;\ TP'(300) = 720 - 2.8(300) = -$$

3) Assembling this information in a table we obtain

			Max		
x	0		257.14		∞
$TP(x)$		↗ ↗ ↗	→	↘ ↘ ↘	
$TP'(x)$		+ + + + +	0	- - - - -	

so that it is clear that total profit has a maximum value when the production level is 257.14. Note that unless fractions of units can be produced, this really means that either 257 or 258 units should be produced. To tell which, one should compute total profit at each of these values. That is, no matter what the mathematical model tells us, in the end "common sense" is also important. Note that if one does such problems for a web based assignment system, such systems fail to have common sense, and so one must use the "dumb" answer supplied by the mathematical model.

13. A manufacturer has been selling 1450 television sets a week at $480 each. A market survey indicates that for each $24 rebate offered to a buyer, the number of sets sold will increase by 240 per week.
    a. Assuming that demand is linear, determine the demand function $p(x)$, where x is the number of television sets sold per week: $p(x) =$ **$-0.1x + 625$**
    b. How large a rebate should the company offer in order to maximize its revenue? **$167.50**
    c. If the weekly cost function is $C(x) = 116{,}000 + 160x$, how should the company set the size of the rebate to maximize profit? **$87.50**

Solution:

a. The demand function, which we are told is linear, passes through the point (1450,480) and has a slope of (-24/240) = -0.1 (rise/run

Appendix E - Common Assignment Problems

was used). Using the point-slope formula we determine that:

$$\frac{p(x)-480}{x-1450}=-\frac{1}{10}\Leftrightarrow p(x)-480=-\frac{1}{10}(x-1450)\Leftrightarrow p(x)=-\frac{1}{10}x+625$$

b. Total revenue is the product price per item times the number of items, so $TR(x) = (-0.1x + 625)x = -0.1x^2 + 625x$. We can assume that $x \in (0,\infty)$.

1) $TR'(x) = -0.1(2x) + 625 = -0.2x + 625$, so $TR'(x) = 0 \Leftrightarrow -0.2x + 625 = 0 \Leftrightarrow x = 3125$
2) There are two intervals of interest, (0,3125) and (3125,∞). We will use 1 to represent the first interval and 4000 to represent the second interval.

$TR'(1) = -0.2(1) + 625 = +$; $TR'(4000) = -0.2(4000) + 625 = -$

			Max		
x	0		3125		∞
TR(*x*)		↗ ↗ ↗	→	↘ ↘ ↘	
TR′(*x*)		+ + + + +	0	- - - - -	

so that it is clear that total revenue has a maximum value when the production level is 3125. At this production level the price would be $p(3125) = (-0.1(3125) + 625) = 312.5$. Since the original price was \$480, the rebate that yields the maximum total revenue is \$480 - \$312.50 = \$167.50.

c) Total profit is the difference between total revenue and total cost. Thus, in this example

$$TP(x) = (-0.1x^2 + 625x) - (116{,}000 + 160x) = -0.1x^2 + 625x - 116{,}000 - 160x = -0.1x^2 + 465x - 116{,}000$$

We can assume that $x \in (0,\infty)$.
$TP'(x) = -0.1(2x) + 465(1) - 0 = -0.2x + 465$. Thus, $TP'(x) = 0 \Leftrightarrow -0.2x + 465 = 0 \Leftrightarrow x = 2{,}325$
There are two intervals of interest, (0,2,325) and (2,325,∞). We will use 1 to represent the first interval and 3000 to represent the second interval.

$TP'(1) = -0.2(1) + 465 = +$; $TR'(3000) = -0.2(3000) + 465 = -$

			Max		
x	0		2,325		∞
TR(*x*)		↗ ↗ ↗	→	↘ ↘ ↘	
TR′(*x*)		+ + + + +	0	- - - - -	

so that it is clear that total profit has a maximum value when the production level is 2,325. At this production level the price would be $p(2{,}325) = (-0.1(2{,}325) + 625) = 392.5$. Since the original price was \$480, the rebate that yields the maximum total profit is \$480 - \$392.50 = \$87.50.

14. A baseball team plays in a stadium that hold 54,000 spectators. With the ticket price at \$40 the average attendance has been 24,000. When the price dropped by \$10, the average attendance rose to 30,000.

a. Assuming that demand is linear, determine the demand function $p(x)$, where x is the number of spectators: $p(x) = \frac{-x}{600}+80$

b. At what price p should the tickets be set so as to maximize revenue? **\$40**

Solution:

a. We are told that $p(x)$ is linear, that (24,000, \$40) is a point on that line, and that (30,000, \$30) is another point on the line. Using the two-point formula for the equation of a line we compute:

Appendix E - Common Assignment Problems

$$\frac{p(x)-40}{x-24000}=\frac{\overset{-10}{\cancel{30-40}}}{\underset{6000}{\cancel{30000-24000}}}\Leftrightarrow p(x)-40=\frac{-1}{600}(x-24000)\Leftrightarrow p(x)=\frac{-x}{600}+80$$

b. Total revenue equals price per ticket times the number of tickets sold. That is, $TR(x)=\left(\frac{-x}{600}+80\right)x=\frac{-x^2}{600}+80x$. Thus,

$$TR'(x)=\frac{-(2x)}{600}+80(1)=80-\frac{x}{300}\ .$$

$$TR'(x)=0\Leftrightarrow 80-\frac{x}{300}=0\Leftrightarrow x=24{,}000$$

In this problem we can see that $x \in [0, 54{,}000]$. Since the total revenue function is a quadratic function, hence continuous everywhere, and we have a closed interval, we now compute the value of total revenue at the critical number, 24,000, and the two interval endpoints, 0 and 54,000.

$$TR(0)=\frac{-(0)^2}{600}+80(0)=0$$

$$TR(24{,}000)=\frac{-(24{,}000)^2}{600}+80(24{,}000)=960{,}000$$

$$TR(54{,}000)=\frac{-(54{,}000)^2}{600}+80(54{,}000)=-54{,}000$$

Clearly the maximum revenue is \$960,000, which comes when attendance is 24,000, gained by setting price at

$$p(24{,}000)=\frac{-(24{,}000)}{600}+80=40\ .$$

15. The manager of a large apartment complex knows from experience that 120 units will be occupied if the rent is \$768 per month. A market survey suggests that, on the average, one additional unit will remain vacant for each \$24 increase in the rent. Similarly, one additional unit will be occupied for each \$24 decrease in the rent. What rent should the manager charge to maximize revenue? **\$1,824**

Solution:

Total revenue is the rent per unit times the number of rented units. From the problem information, if we let x represent the number of rented units, then rent, as a function of x, is a linear function passing through (120,\$768) and a slope of 24/(-1) = -24. Using the point-slope formula for a straight line:

$$\frac{p(x)-768}{x-120}=-24\Leftrightarrow p(x)-768=-24(x-120)\Leftrightarrow p(x)=-24x+3{,}648$$

Thus, total revenue is $TR(x) = p(x)x = (-24x + 3{,}648)x = -24x^2 + 3{,}648x$. All we know is that $x \in (0,\infty)$ since we were given no further information. Although TR is a continuous function, the interval is not closed, so we must resort to checking the shape of the graph to see if there is a global maximum revenue on that interval, and if so, at what value of x.

We see that $TR'(x) = -48x + 3{,}648$, so $TR'(x) = 0 \Leftrightarrow -48x + 3{,}648 = 0 \Leftrightarrow x = 76$.

Thus there are two intervals of interest, $(0,76)$ and $(76,\infty)$. We will use 1 to represent the first interval and 100 to represent the second interval.

$TR'(1) = -48(1) + 3{,}648 = +$; $TR'(100) = -48(100) + 3{,}648 = -$

			Max		
x	0		76		∞
$TR(x)$		↗ ↗ ↗	→	↘ ↘ ↘	
$TR'(x)$		+ + + + +	0	- - - - -	

so that it is clear that total revenue has a maximum value when the number of apartments rented is 76. At this occupancy level the price would be $p(76) = (-24(76) + 3{,}648) = \$1{,}824$.

Appendix F - Student Consent Form

A Study of the Factors Influencing Success in Mathematics Amongst CÉGEP Students

Directions to the Student

A team of researchers at Vanier College is investigating the reasons why students succeed/do not succeed when studying mathematics. We would like you to participate in this research. This will involve taking an Algebra skills diagnostic test and by allowing the college registrar/ministry of education to provide us with information that is in your file. If you are interested in more information, or the results of this research, please contact the project director, Helena Dedic, by telephone at 744-7500-2-7016, or by e-mail at dedich@vaniercollege.qc.ca.

I, the undersigned, consent to participate with the assurance that all data will be kept **confidential** and that this will in **no way affect my academic record at CÉGEP**. I understand that I have the right to refuse to participate at any time, and that such refusal will also in no way affect my academic record at CÉGEP. Further, I understand that should I decide to participate at this time, I can subsequently change my mind by sending an e-mail to the project director, **Helena Dedic**, at **dedich@vaniercollege.qc.ca**, informing her of my decision. In such a circumstance, all data that I have contributed will be withdrawn and my decision will also in no way affect my academic record at CÉGEP.

DATE: ______________________________

PRINT NAME: ______________________________ | ______________________________

Given Name(s) Family Name

STUDENT #: ______________________________

SIGNATURE: ____